Optimal strokes for axisymmetric microswimmers

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1 Introduction

There is a growing interest in understanding optimal swimming strategies for natural and artificial microswimmers [1–4]. For engineered systems, the goal is to design microrobots capable of efficient self-propulsion [5–7]. For micro-organisms, a natural question is whether their swimming patterns may be optimal in some sense, and whether they have been selected by evolutionary pressure [8,9].

Small size leads to low-Reynolds-number (Re) flows, where all inertial effects are negligible and the fluid is governed by steady Stokes equations [10]. This flow regime, when looked at from the perspective of our own swimming experience, may often seem paradoxical. In fact, while fish or men propel themselves by accelerating the surrounding water, microswimmers cannot rely on inertia. Thus, they swim by exploiting the viscous resistance of water. Moreover, reciprocal swimming strategies that work well in the inertial regime (e.g., wagging a rigid fin, or opening and closing the two rigid valves of a scallop) are completely ineffective at low Re. Indeed, because of the linearity of Stokes equations and their symmetry under time reversals, whatever forward motion is produced in a part of a reciprocal stroke, it will be exactly cancelled by a backward motion when the same part of the stroke is executed backwards. The crucial obstacle to be overcome in inertialess swimming is how to induce, via time-periodic shape changes, positional changes that are not time periodic.

The special case of axisymmetric swimmers provides an interesting balance between complexity and generality of the attainable results. This paper presents a complete analysis of axisymmetric swimmers whose shape depends on finitely many parameters and a general method to determine strokes of maximal efficiency. The main feature qualifying our approach is the possibility of resolving hydrodynamic interactions arising from the swimmer motion in their full complexity, without being confined to asymptotic regimes in which explicit formulas for hydrodynamic forces are available.

2 The mathematics of swimming

Consider a swimmer having an axisymmetric shape Ω with axis of symmetry parallel to the unit vector i and swimming along i. The state s of the system is described by N + 1 scalar configuration parameters: s = (x(1),...,x(N+1)). Alternatively, s can be specified by a position c, e.g., the coordinate of the center of mass along the symmetry axis and by N shape parameters ξ = (ξ1,...,ξN). This change of coordinates is invertible and the generalized velocities u(i) = ˙x(i) are linear functions of the time derivatives of position and shape:

\[ (u(1),...,u(N+1))^t = A(\xi)(\dot{\xi}_1,...,\dot{\xi}_N,\dot{c})^t. \] (1)

The entries of the (N + 1) × (N + 1) matrix A in (1) are independent of c by translational invariance.

Swimming describes the ability to change position in the absence of external propulsive forces by executing a cyclic shape change. We will confine attention to cases in which cyclic shape changes are described by N real-valued time-periodic shape functions \( t \mapsto \xi_i(t) \in \mathbb{R}, i = 1,...,N \). Since the swimmer’s inertia is being neglected, the total
drag force exerted by the fluid on the swimmer is equal and opposite to the total propulsive force and must also vanish. All force components in directions perpendicular to \( \vec{c} \) vanish by symmetry and self-propulsion is expressed simply by

\[
0 = \int_{\partial \Omega} \sigma \vec{n} \, dS \cdot \vec{i},
\]

where \( \sigma \) is the stress tensor in the fluid surrounding \( \Omega \), and \( \vec{n} \) is the outward unit normal to \( \partial \Omega \), the boundary of \( \Omega \). The stress \( \sigma_{\alpha\beta} = \eta (\partial v_\alpha / \partial x_\beta + \partial v_\beta / \partial x_\alpha) - p \delta_{\alpha\beta} \) (here \( \eta \), \( v \), and \( p \) are viscosity, velocity, and pressure) is obtained by solving Stokes equation outside \( \Omega \) with prescribed boundary data \( \vec{v} = \vec{v} \) on \( \partial \Omega \). Here \( \vec{v} \) is the velocity at the points on the boundary \( \partial \Omega \) of the swimmer, which, just as in (1), depends linearly on \( \vec{c} \) and \( \vec{e} \),

\[
\vec{v} = \sum_{i=1}^{N} \vec{v}_i(\vec{\xi},x) \dot{\vec{\xi}}_i + \vec{v}_{N+1}(\vec{\xi},x) \dot{\vec{\xi}}, \quad x \in \partial \Omega,
\]

through known functions \( \vec{v}_i \), \( i = 1, \ldots, N+1 \) specifying the kinematics of the swimmer. In particular,

\[
\vec{v}_{N+1}(\vec{\xi},x) \equiv 1, \quad x \in \partial \Omega.
\]

In what follows we will write \( \sigma \) as \( \sigma(\vec{v}) \) whenever we want to emphasize its linear dependence on \( \vec{v} \) and its nonlinear dependence on the shape \( \vec{\xi} \) of \( \Omega \). By linearity of Stokes equations, drag depends linearly on \( \vec{v} \) and hence on \( \vec{\xi} \) and \( \vec{c} \), see (3). We can thus write (2) as

\[
0 = \sum_{i=1}^{N} \varphi_i(\vec{\xi}) \dot{\vec{\xi}}_i + \varphi_{N+1}(\vec{\xi}) \dot{\vec{c}}.
\]

The coefficients \( \varphi_i \) relating drag force to velocities in (5) are independent of \( c \) by translational invariance, and are given by

\[
\varphi_j(\vec{\xi}) = \int_{\partial \Omega} \sigma(\vec{v}[j]) \vec{n} \, dS \cdot \vec{i}, \quad j = 1, \ldots, N+1,
\]

where the functions \( \vec{v}_i \) are given in (3). The coefficient \( \varphi_{N+1} \) of \( \dot{\vec{c}} \) represents the drag force corresponding to a rigid translation along the symmetry axis at unit speed, see (4), and it never vanishes. Thus, (5) can be solved for \( \dot{\vec{c}} \) yielding

\[
\dot{\vec{c}} = \sum_{i=1}^{N} V_i(\xi_1, \ldots, \xi_N) \dot{\vec{\xi}}_i = \vec{V}(\vec{\xi}) \cdot \dot{\vec{\xi}},
\]

where

\[
V_i(\vec{\xi}) = - \frac{\varphi_i(\vec{\xi})}{\varphi_{N+1}}, \quad i = 1, \ldots, N
\]

and the \( \varphi_i \) are given in (6). Equation (7) links shape changes to positional changes through shape-dependent coefficients encoding the hydrodynamic interactions between \( \Omega \) and the surrounding fluid due to shape changes at rate \( \dot{\vec{\xi}} \).

A stroke is a \( T \)-periodic map \( t \mapsto \vec{\xi}(t) \) describing a closed path \( \gamma \) in the space \( S \) of admissible shapes. In view of (7), swimming requires that

\[
0 \neq \Delta c = \int_0^T \sum_{i=1}^N V_i \dot{\vec{\xi}}_i \, dt,
\]

i.e., that the differential form \( \sum_{i=1}^N V_i \, d\vec{\xi}_i \) is not exact.

Following [11], we define swimming efficiency as the inverse ratio between the average power expended by the swimmer during a stroke starting and ending at the shape \( \vec{\xi}_0 \) and the power that an external force would spend to translate the system rigidly at the same average speed \( \dot{c} = \Delta c / T \):

\[
\text{Eff}^{-1} = \frac{1}{T} \int_0^T \int_{\partial \Omega} \sigma \vec{n} \cdot \vec{v} \, dS \, dt \equiv \frac{\int_0^T \int_{\partial \Omega} \sigma \vec{n} \cdot \vec{v} \, dS \, dt}{6 \pi \eta L \Delta c^2}.
\]

Here \( L = L(\vec{\xi}_0) \) is the effective radius of the swimmer, defined as the radius of the sphere giving the same drag force as the one produced by the system when rigidly translating along \( \vec{i} \) with shape \( \vec{\xi}_0 \) and velocity \( \vec{c} \). Time has been rescaled linearly in (11) to obtain \( T = 1 \). The expression in the denominator in (10) and (11) comes from a generalization of Stokes formula giving the drag on a sphere of radius \( L \) moving at velocity \( \vec{c} \) as \( 6 \pi \eta L \dot{c} \). By linearity of Stokes equations and (7), the expended power in the numerator in (11) is a quadratic form in \( \vec{\xi} \) and can be written as

\[
\int_{\partial \Omega} \sigma \vec{n} \cdot \vec{v} \, dS = \sum_{i,j=1}^N G_{ij}(\vec{\xi}) \dot{\vec{\xi}}_i \dot{\vec{\xi}}_j = \vec{G}(\vec{\xi}) \dot{\vec{\xi}} \cdot \dot{\vec{\xi}}.
\]

The entries of the \( N \times N \) symmetric and positive-definite matrix \( \vec{G}(\vec{\xi}) \) are independent of \( c \) by translational invariance. More explicitly

\[
G_{ij}(\vec{\xi}) = \int_{\partial \Omega} \sigma(\vec{\xi}[j] \vec{\xi}[j]) \vec{n} \cdot \vec{\omega}_i \vec{w}_j \, dS, \quad i,j = 1, \ldots, N,
\]

where

\[
\vec{\omega}_i(\vec{\xi},x) = \vec{v}_i(\vec{\xi},x) + V_i(\vec{\xi}) \vec{v}_{N+1}(\vec{\xi},x), \quad x \in \partial \Omega,
\]

while \( \vec{v}_i \) and \( V_i \) are given by equations (3) and (8).

Strokes of maximal efficiency are those producing a given displacement \( \Delta c \) with minimal expended power. From (11), maximal efficiency is obtained by minimizing

\[
\int_0^1 \int_{\partial \Omega} \sigma \vec{n} \cdot \vec{v} \, dS \, dt = \int_0^1 \vec{G}(\vec{\xi}) \dot{\vec{\xi}} \cdot \dot{\vec{\xi}} \, dt
\]

subject to the constraint

\[
\int_0^1 \vec{V}(\vec{\xi}) \cdot \dot{\vec{\xi}} \, dt = \Delta c \neq 0
\]

among all closed curves \( \vec{\xi} : [0,1] \rightarrow S \) in the set \( S \) of admissible shapes such that \( \vec{\xi}(1) = \vec{\xi}(0) \). The Euler-Lagrange