The construction of Markov processes in random environments and the equivalence theorems

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Abstract In sec. 1, we introduce several basic concepts such as random transition function, \( p - m \) process and Markov process in random environment and give some examples to construct a random transition function from a non-homogeneous density function. In sec. 2, we construct the Markov process in random environment and skew product Markov process by \( p - m \) process and investigate the properties of Markov process in random environment and the original process and environment process and skew product process. In sec. 3, we give several equivalence theorems on Markov process in random environment.

Keywords: random transition function, \( p - m \) process, Markov process in random environment, skew product Markov process, density function.

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The Markov chains in random environments have been pursued for some time. Nawrotzki\(^1\) established a general theory of this topic. Cogburn\(^{2-5}\) developed a theory in a wider context by making use of some more powerful tools such as the theory of Hopf Markov chains. He obtained many important results on this field, such as the classification of state, invariance measure, ergodic limit theorems and central limit theorems. Orey\(^6\) reviewed the works on this field, especially the works by Cogburn and also gave some new results and open problems in a specially invited paper. Recently Kifer\(^7\) studied the limit theorems for random transformations. His results can be considered as a generalization of Cogburn’s work in ref. \([5]\) in a sense. Li\(^8\) investigated the recurrence and invariant measure for Markov chains in bi-infinite environment.

The objects of all of the above works are Markov chains in random environments, and the set of time parameter is discrete. Now in our paper the object is Markov process in random environment. The sets of parameters of original process and environment are continuous. In sec. 1 we introduce the concept of random transition function and construct it from non-homogeneous density function. In sec. 2 we construct the Markov process in random environment with given random transition function and investigate the properties of such process. In sec. 3 we give several equivalence theorems on Markov processes in random environments. For classical Markov chains, see refs. \([9-12]\).
1 Definitions and examples

Let \( (X, \mathcal{A}) \) and \( (\Theta, \mathcal{B}) \) be two abstract measurable spaces, \( \mathbb{R} = (-\infty, \infty), \mathbb{R}_+ = [0, \infty), \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}, \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \). Let \( D \) be the collection of all Borel sets in \( D, X^* = X^{D \cap \mathbb{R}_+}, \mathcal{A}^* = \mathcal{A}^{D \cap \mathbb{R}_+} \). Let \( \Theta^* \) be a collection of functions defined on \( D \) and taking values in \( \Theta \). Let \( \theta^*[s, s + t] \) be the restriction of \( \theta^* \) on \( [s, s + t) \) for any \( \theta^* \in \Theta^* \), \( \Theta^*[s, s + t] = \{ \theta^*[s, s + t] : \theta^* \in \Theta^* \} \).

**Definition 1.1.** Let \( p(\cdot, \cdot) : \Theta^*(re) \times X \times \mathcal{A} \to [0, 1] \). If

1. \( p(\theta^*[s, s + t]; x, \cdot) \) is a measure for any fixed \( \theta^*[s, s + t] \in \Theta^*(re) \) and \( x \in X \),
2. \( p(\theta^*[s, s + t]; x, \cdot) \) is measurable w.r.t. \( (\mathcal{B}^{D \cap [s, s + t)}, \mathcal{A}) \) as a function of \( (\theta^*[s, s + t], x) \) for any fixed \( s \geq 0, t > 0 \) and \( A \in \mathcal{A} \),
3. \( p \) satisfies the random Kolmogorov-Chapmann equation:

\[
p(\theta^*[s, s + t + u]; x, A) = \int_X p(\theta^*[s, s + t]; x, dy)p(\theta^*[s + t, s + t + u]; y, A), \quad (1.1)
\]

\( \theta^* \in \Theta^* \), \( s \geq 0, t, u > 0, x \in X, A \in \mathcal{A} \), then we call \( p \) a random sub-transition function in \( \Theta^* \) (R.S.T.F). An R.S.T.F. \( p \) is said to be a random transition function (R.T.F.), if \( p(\theta^*[s, s + t]; x, X) \equiv 1 \).

**Definition 1.2.** Let \( p(\cdot, \cdot) : X \times \mathcal{A} \to [0, 1] \). If \( p(\cdot, A) \) is measurable w.r.t. \( \mathcal{A} \) for any fixed \( A \in \mathcal{A} \) and \( p(x, \cdot) \) is a measure for any fixed \( x \in X \), then we call \( p \) a sub-Markov kernel (S.M.K.). A S.M.K. \( p \) is called a Markov kernel (M.K.), if \( p(x, X) \equiv 1 \).

**Definition 1.3.** Let \( (X, \mathcal{A}), (\Theta, \mathcal{B}), D, \Theta^*, \mathcal{B}^* \) be defined as before, \( m \) be a probability measure on \( \mathcal{B}^* \), and \( p(\theta^*[s, s + t]; x, A) \) a random transition function in \( \Theta^* \). Then we call \( V = (X, \mathcal{A}; \Theta, \mathcal{B}; p, m) \) a \( p - m \) process on \( (D, \Theta^*) \). Especially we call \( V \) a \( p - m \) chain in \( \Theta^* \) if \( D = \mathbb{Z} \) and a \( p - m \) process in \( \Theta^* \) if \( D = \mathbb{R} \).

If \( \Theta^* = \Theta^D \) is the collection of all functions defined on \( D \) and taking values in \( \Theta \), the words “in \( \Theta^* \)” will be omitted.

**Definition 1.4.** Let \( X^* = \{ X_t, t \in D \cap \mathbb{R}_+ \} \) and \( \xi^* = \{ \xi_t, t \in D \} \) be two processes on probability space \( (\Omega, \mathcal{F}, P) \) taking values in \( X \) and \( \Theta \) respectively, and let \( p(\theta^*[s, s + t]; x, A) \) a random transition function. If

\[
(M) : P(X_{s+t} \in A, \xi_{s+t} = B | \xi_u, u \in D \cap [0, s], \xi_v, v \in D \cap (-\infty, s + t))
\]

\[
= p(\xi^*[s, s + t]; X_s, A)P(\xi_{s+t} = B | \xi_v, v \in D \cap (-\infty, s + t)) \quad \text{a.s.} \quad (1.2)
\]

\( A \in \mathcal{A}, B \in \mathcal{B}, t > 0, s, s + t \in D \cap \mathbb{R}_+ \), then we call \((X^*, \xi^*)\) a Markov process.

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