Superconvergence and asymptotic expansions for linear finite element approximations on criss-cross mesh

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Abstract In this paper, we discuss the error estimation of the linear finite element solution on criss-cross mesh. Using space orthogonal decomposition techniques, we obtain an asymptotic expansion and superconvergence results of the finite element solution. We first prove that the asymptotic expansion has different forms on the two kinds of nodes and then derive a high accuracy combination formula of the approximate derivatives.

Keywords: criss-cross mesh, linear finite element, superconvergence.

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1 Introduction

In recent years, since the discovery of the close relationship between high accuracy post-processing and posteriori error estimate, there has been growing interest in the superconvergence and other kinds of high accuracy methods such as defect correction and extrapolation. Roughly speaking, there are three kinds of approaches to investigating superconvergence properties of finite element methods. One is the Chinese approach based on "element analysis" (see refs. [1—4]), with which one can get exact information on the error and derive asymptotic expansions. Moreover, one can prove the global results. It depends on some uniform (with perturbation) properties of the meshes. The second approach was based on the "local symmetry" of the mesh. It can deal with more general meshes but it can only enable us to get interior superconvergence[5,6]. The third one is based on numerical validation[7,8]. Since the criss-cross mesh is not a uniform mesh in the sense of 3-directional parallel, as a typical type of grid partition (other typical types are Chevron mesh, Union Jack mesh, etc.), it is more difficult to study the superconvergence and asymptotic expansion of linear element on the criss-cross mesh. So far, rare are the results of superconvergence on non-uniform mesh[5,6].

In this paper, we consider the following model problem

\[
\begin{align*}
-\Delta u &= f, & (x,y) \in \Omega &= [0,1] \times [0,1], \\
u(x,y) &= 0, & (x,y) \in \partial \Omega,
\end{align*}
\]

(1.1)
where $f(x, y)$ is a known real-valued function, which is assumed to be smooth enough to ensure that the problem (1.1) has a unique solution in some Sobolev space.

In the following, let $\mathcal{O}^h$ be the criss-cross partition of the domain $\mathcal{O}$, as shown in fig. 1(a), and let $\mathcal{P}_k$ be the set of polynomials of degree not more than $k$, where $h$ is the maximal diameter of all the partition elements in $\mathcal{O}^h$.

Fig. 1. (a) The criss-cross mesh; (b) the standard quadrangle cell.

Linear finite element space $V_h$ on criss-cross mesh is defined by

$$V_h = \{v_h(x, y) : v_h(x, y) \in C(\bar{\Omega}) \cap H^1_0(\Omega), v_h|_T \in \mathcal{P}_1, \forall T \in \mathcal{O}^h\}.$$ 

The finite element solution of equation (1.1) $u_h \in V_h$ satisfies

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (1.2)$$

where $a(u, v) = \int_\Omega (u_xv_x + u_yv_y)dxdy$, $(f, u) = \int_\Omega fudx$.

Because of the special structure of the criss-cross mesh $\mathcal{O}^h$, i.e. the support of the interpolation base functions corresponding to the nodes is contained in a square element, it is possible to eliminate certain variables locally, meaning that, by using proper linear combinations of the interpolation base functions in $V_h$, the space $V_h$ can be orthogonally decomposed into two subspaces. One only plays a conforming role and the other plays an essential role in the finite element systems. Based on those analysis, we can get superconvergence results and asymptotic expansion of the linear element solution $u_h$ on the criss-cross mesh. We point out that the methods and techniques of analysis proposed in this paper can be extended to more general second order elliptic equations with non-uniform mesh.

2 Some lemmas

Let $h_x$ and $h_y$ be the mesh sizes, and let $N_x$ and $N_y$ be the partition numbers in $x$-direction and $y$-direction respectively. For simplicity, we denote the nodal coordinate $(kh_x, lh_y)$ by $(k, l)$ in this paper. There are two patterns of interior nodes in the partition of $\mathcal{O}$.

$$S_1 = \{(k, l) | k, l \text{ are integers, } 1 \leq k \leq N_x - 1 \text{ and } 1 \leq l \leq N_y - 1\},$$

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