THE GENERAL RELATIVISTIC AND COVARIANT FORM OF THE I. HELMHOLTZ VORTICITY THEOREM AND GEOPHYSICAL APPLICATION

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The general relativistic and covariant differential form of Helmholtz’s first vorticity theorem is presented. We prove in relation with it an invariant kinematic identity which is the generalisation of the Helmholtz theorem for general continua.

Keywords: geophysical hydrodynamics; Helmholtz vorticity theorem; invariant kinematic identity

The general-relativistic and with it, the general-covariant formulation of the equations of the mechanics of continua has a significant advantage, namely that the so formulated equations are automatically valid in arbitrary systems of coordinates, but also in arbitrary systems of reference. The studied continuum may correspondingly be exposed to arbitrary inertial forces; the effect of these inertial forces (especially the centrifugal and the Coriolis forces) on the continuum is through the general relativistic covariance automatically taken into account. Moreover, the general-relativistic form also contains in the case of not disappearing Riemannian tensor $R^r_{\beta\gamma\delta}$ the effect of the gravity field on the continuum. Thus the effect of weak Newtonian gravity fields on the continuum is included for the limiting case of weak static gravity fields. Finally the general-relativistic form contains all special-relativistic effects, too and therefore it generalises the consequences of the classical mechanics of continua on velocities which are comparable to the velocity of light.

Additionally the covariant tensor calculus is especially elegant and simple, and as a consequence, it can describe complicated physical situations of a high generality with few symbolic operations.

We use in the following the tensor calculus with the symbolism used by Einstein (1969). Comma means the usual partial derivative after the corresponding index:

$$\Phi,\lambda = \frac{\partial}{\partial x^\lambda} \Phi.$$

The semicolon means the corresponding covariant derivative, thus e.g. for a vector field $A^\alpha$:

$$A^\alpha;\lambda = A^\alpha,\lambda + \Gamma^\alpha_{\beta\lambda} A^\beta.$$

where $\Gamma^\alpha_{\beta\lambda},$ is Christoffel’s three-index-symbol:

$$\Gamma^\alpha_{\beta\lambda} = \frac{1}{2} g^{\alpha\tau} (-g_{\beta\lambda,\tau} + g_{\lambda,\tau} - g_{\tau,\beta\lambda})$$

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and $g_{\alpha \beta}$ is the metric fundamental tensor and it is

$$A_{\mu;\lambda} - A_{\lambda;\mu} = A_{\mu;\lambda} - A_{\lambda;\mu}.$$

According to Einstein, one has to sum after the double indices. The Greek indices run from zero to three, the Latin ones from 1 to 3, the sign is fixed so that time-like vectors have a negative square of the absolute value (the inertial index is $-1, +1, +1, +1$). Latin indices are generally three-dimensional spatial indices.

We define the four-vector of the velocity of a particle of the continuum according to:

$$u^{\alpha} = \frac{dx^{\alpha}}{dt} \quad (1a)$$

where the following expression is:

$$dt = \frac{1}{c} \sqrt{-g_{\alpha \beta} dx^{\alpha} dx^{\beta}} \quad (1b)$$

(where $c$ is the velocity of light in the inertial system) the differential of the eigen-time. The four-velocity $u^{\alpha}$ is then normed as

$$g_{\alpha \beta} u^{\alpha} u^{\beta} = u^{\alpha} u_{\alpha} = -c^2. \quad (2)$$

The velocity $u^{\alpha}$ is a contravariant vector to which the co-vector

$$u_{\beta} = g_{\beta \alpha} u^{\alpha} \quad (3)$$

belongs. Using this co-vector, we define the antisymmetric tensor of the rotation of the continuum:

$$\omega^{\alpha}_{\beta} = u^{\alpha}_{\beta} - u^{\beta}_{\alpha} \quad (4)$$

We understand as the total (material) derivative of a covariant quantity $\Phi$ after the eigen-time $\tau$ the following:

$$\frac{D}{Dt} \Phi = \Phi_{,\lambda} u^{\lambda} \quad (5)$$

Without any supposition about the dynamics and structure of the continuum we find a simple expression for the total temporal variation $\frac{D}{Dt} \omega^{\alpha}_{\beta}$ of the rotation $\omega^{\alpha}_{\beta}$: it is at first valid that

$$\frac{D}{Dt} \omega^{\alpha}_{\beta} = \omega^{\alpha}_{\beta;\lambda} u^{\lambda} = u^{\alpha}_{\beta;\lambda} u^{\lambda} - u^{\beta}_{\beta;\lambda} u^{\lambda} \quad (6)$$

and by changing the sequence of differentiation we obtain:

$$\frac{D}{Dt} \omega^{\alpha}_{\beta} = (u^{\alpha;\lambda}_{\beta} - u^{\beta;\lambda}_{\alpha}) u^{\lambda} - (R^{\kappa;\alpha}_{\beta;\lambda} - R^{\kappa;\beta}_{\alpha;\lambda}) u^{\kappa} u^{\lambda} \quad (7)$$

Here $R^{\kappa;\alpha}_{\beta;\lambda}$ is the great Riemann-tensor.

On the base of the symmetry property of this tensor, the second term on the right side of Eq. (7) disappears. The first term is transformed using the identity:

$$u^{\alpha;\lambda}_{\beta} u^{\lambda} = -u^{\alpha}_{\beta;\lambda} u^{\lambda} \quad (8a)$$

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