Uncertainty relations expressed by Shannon-like entropies

V. Majerník\textsuperscript{1,2}, Eva Majerníková\textsuperscript{1,3}, S. Shpyrko\textsuperscript{3}

\textsuperscript{1} Department of Theoretical Physics, Palacký University, Tř. 17. listopadu 50, CZ-77207 Olomouc, Czech Republic
\textsuperscript{2} Institute of Mathematics, Slovak Academy of Sciences, Štefánikova 49, SK-81473 Bratislava, Slovak Republic,
\textsuperscript{3} Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta, SK-84 228 Bratislava, Slovak Republic

Received 6 November 2002; revised 22 April 2003

Abstract: Besides the well-known Shannon entropy, there is a set of Shannon-like entropies which have applications in statistical and quantum physics. These entropies are functions of certain parameters and converge toward Shannon entropy when these parameters approach the value 1. We describe briefly the most important Shannon-like entropies and present their graphical representations. Their graphs look almost identical, though by superimposing them it appears that they are distinct and characteristic of each Shannon-like entropy. We try to formulate the alternative entropic uncertainty relations by means of the Shannon-like entropies and show that all of them equally well express the uncertainty principle of quantum physics.

Keywords: Entropic uncertainty relations, quantum theory, probabilistic systems, generalized entropies, Tsallis entropies.
PACS (2000): 03.67.-a, 05.20.-y, 89.70.+c

1 Introduction

All types of the Shannon-like (S-L, for short) entropies\textsuperscript{\S}, like the Shannon entropy, are based on the notions of probability and uncertainty. Although there is a well-defined mathematical theory of probability, there is no universal agreement about the meaning

\S The Shannon-like entropies are sometimes called the nonstandard [16] or generalized entropies [23].
of probability. Thus, for example, there is the view that probability is an objective property of a system, and another view that it describes a subjective state of belief of a person. Then there is the also the point of view that the probability of an event is the relative frequency of its occurrence in a long or infinite sequence of trials. This latter interpretation is often employed in the mathematical statistics and statistical physics. The probability in everyday life means the degree of ignorance about the outcome of a random trial. Commonly, the probability is interpreted as the degree of the subjective expectation of an outcome of a random trial. Both subjective and statistical probability are “normed” which means that the degree of expectation that an outcome of a random trial occurs, and the degree of the “complementary” expectation, that it does not, always add up to unity.

Intuitively, the uncertainty of a random trial is given by the spread of probabilities of its outcomes. The uncertainty of a many component probability distribution is quantitatively given by one number $H$ that is a function of all components of a probability distribution, $H(P) = F(P_1, P_2, ..., P_n)$. This number $H$ satisfies the following requirements:

(i) If the probability distribution contains only one component then $H(P) = 0$. In this case, there is no uncertainty in a random trial because one outcome is realized with certainty.

(ii) The more spread the probability distribution $P$ is, the larger becomes the value of its uncertainty.

(iii) For a uniform probability distribution $P_u$, $H(P_u)$ becomes maximal.

An important quantity in the theory of probability is the random variable. A random variable $\tilde{x}$ is a mathematical quantity assuming a set of values with corresponding probabilities. All data necessary for the characterization of a random trial, and the assigned random variable, are usually given by a so-called probabilistic scheme. If $\tilde{x}$ is a discrete random variable then its probability scheme is of the form

<table>
<thead>
<tr>
<th>S</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>...</th>
<th>$S_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>$P(x_1)$</td>
<td>$P(x_2)$</td>
<td>...</td>
<td>$P(x_n)$</td>
</tr>
<tr>
<td>X</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>...</td>
<td>$x_n$</td>
</tr>
</tbody>
</table>

$S_1, S_2, ..., S_3$ are the outcomes of a random trial (in quantum physics the quantum states), $P(x_1), P(x_2), ..., P(x_n)$ are their probabilities and $x_1, x_2, ..., x_3$ are the values defined on $S_1, S_2, ..., S_n$ (in quantum physics the eigenvalues). A probability distribution, $P = \{P_1, P_2, ..., P_n\}$, is the complete set of probabilities of all individual outcomes of a random trial.

It is well-known that there are several measures of the uncertainty in the theory of probability which can be divided into two classes [29]:

---

From the mathematical point of view, the probabilistic uncertainty measures map the nonnegative orthant $\mathbb{R}^n_+$ of the n-dimensional Euclidean space $\mathbb{R}^n$ into $\mathbb{R}$. 