A numerically stable least squares solution to the quadratic programming problem

E. Õubi*

Department of Economics, Tallinn University of Technology, Kopli 101, 11712 Tallinn, Estonia

Received 31 May 2007; accepted 10 November 2007

Abstract: The strictly convex quadratic programming problem is transformed to the least distance problem - finding the solution of minimum norm to the system of linear inequalities. This problem is equivalent to the linear least squares problem on the positive orthant. It is solved using orthogonal transformations, which are memorized as products. Like in the revised simplex method, an auxiliary matrix is used for computations. Compared to the modified-simplex type methods, the presented dual algorithm QPLS requires less storage and solves ill-conditioned problems more precisely. The algorithm is illustrated by some difficult problems.

MSC: 90C05, 65K05

Keywords: Quadratic programming • system of linear inequalities • method of least squares • Householder transformation • successive projection

© Versita Warsaw and Springer-Verlag Berlin Heidelberg.

1. Introduction

The purpose of the paper is to solve a quadratic programming problem using the highly developed least squares technique. This method is used not only in mathematics but also in statistics, physics etc., where mainly nonlinear problems are solved by composing a certain number of similar linear least squares problems, differing in a variable or constraint. In this paper it will be proved that such a methodology can be used also for solving quadratic programming problems. The least squares technique and its applications to the mathematical programming are described thoroughly in [3], [7].

Several numerous algorithms have been proposed for solving quadratic programming problems (Wolfe, Beale, Barankin - Dorfman, Dantzig, Goldfarb, Powell, Stoor, Lawson-Hanson), see [3], [2], [5], [1]. The above methods can for the most part be categorized as either modified-simplex type methods or projection methods. The former perform simplex type pivots on basis matrices or tableaux of size \((m+n)\) that are derived from the Kuhn-Tucker optimality conditions for the QPP (1). The latter are based upon projections onto active sets of constraints and employ operators of size no larger than \((n \times n)\). Consequently, projection methods are usually more efficient and require less storage than methods of the modified simplex type. In this paper we present a projection-type dual algorithm for solving QPP.

* E-mail: evaldy@tv.ttu.ee
Let us have a quadratic programming problem

\[
\min \{0.5(y, By) + (d, y)\} \tag{1}
\]

\[Cy \leq h,\]

where \(B\) is a positive-definite \(n \times n\) matrix, \(G\) is an \(m \times n\) matrix, \(y \in \mathbb{R}^n, d \in \mathbb{R}^n, h \in \mathbb{R}^m, (,\) stands for the inner product in the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\), see [4].

The vector of variables \(y\) may also be subject to equality constraints

\[Cy = g.\]

In Section 3 we transform these constraints into inequalities. Using the Choleski decomposition with \(B = D^TD\) and shifting

\[y = D^{-1}x - B^{-1}d, \tag{2}\]

problem (1) is transformed into the following:

\[
\min \{z = 0.5 \| x \|^2 \} \tag{3}
\]

\[Ax \leq b,\]

where \(A = GD^{-1}\) and \(b = h + GB^{-1}d\), \(\|\cdot\|\) stands for the Euclidean norm in \(\mathbb{R}^n\). As proved in [7], such a quadratic programming problem is equivalent to the least squares problem

\[
A^Tu = 0, \tag{4}
\]

\[(b, u) = -1\]

\[u \geq 0\]

or

\[
\min\{\varphi(u) = \| A^Tu \|^2 + (1 + (b, u))^2\} \tag{5}
\]

\[u \geq 0,\]

where \(u \in \mathbb{R}^m\). The solution of minimum norm to the problem (3) is

\[
\hat{x} = \frac{-A^T \hat{u}}{1 + (b, \hat{u})}, \tag{6}
\]

where \(\hat{u}\) is the least squares solution to the problem (5).

A detailed description of the algorithm QPLS with examples is given in Section 2. Here we shall briefly discuss the solution of the least squares problem problem (5) for which two finite orthogonal methods are given in papers [7], [6].

In the first one, like in the simplex method, at each step all the coefficients of the system (4) are transformed. Such an algorithm is used for problems of small dimensions only. The second method is similar to the revised simplex method, see [6]. The matrix of the system (4) is not transformed in the process of computations. For this matrix the \(QR\)-transformation is used, \(Q\) is an orthogonal and \(R\) an upper-triangular matrix.

In Section 2 the algorithm QPLS is described and an example is given. In Section 3 equality constraints are transformed into inequalities. In Section 4 some test problems are solved.