Critical configurations of planar robot arms

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Abstract: It is known that a closed polygon $P$ is a critical point of the oriented area function if and only if $P$ is a cyclic polygon, that is, $P$ can be inscribed in a circle. Moreover, there is a short formula for the Morse index. Going further in this direction, we extend these results to the case of open polygonal chains, or robot arms. We introduce the notion of the oriented area for an open polygonal chain, prove that critical points are exactly the cyclic configurations with antipodal endpoints and derive a formula for the Morse index of a critical configuration.

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1. Introduction

Geometry of various special configurations of robot arms modeled by open polygonal chains appears essential in many problems of mechanics, robot engineering and control theory. The present paper is concerned with certain planar configurations of robot arms appearing as critical points of the oriented area considered as a function on the moduli...
space of the arm in question. This setting naturally arose in the framework of a general approach to extremal problems on configuration spaces of mechanical linkages developed in [2, 4, 5], which has led to a number of new results on the geometry of cyclic polygons [3, 6] and suggested a variety of open problems. The approach and results of [2, 5] provided a paradigm and basis for the developments presented in this paper.

Let us now outline the structure and main results of the paper. We begin with recalling necessary definitions and basic results concerned with moduli spaces and cyclic configurations. In Section 3 we prove that critical configurations of a planar robot arm are given by the cyclic configurations with diametrical endpoints called diacyclic, Theorem 3.1, and describe the structure of all cyclic configurations of a robot arm, Theorem 3.3. Next, we establish that, for a generic collection of lengths of the links, the oriented area is a Morse function on the moduli space, Theorem 3.6, and provide some explications in the case of a 3-arm. In Section 4 we prove an explicit formula for the Morse index of a diacyclic configuration, Theorem 4.6, and illustrate it by a few visual examples. In conclusion we briefly discuss several open problems and related topics.

2. Oriented area function for planar robot arm

Let \( L = (l_1, \ldots, l_n) \), \( L \in \mathbb{R}_n \). Informally, a robot arm, or an open polygonal chain is defined as a linkage built up from rigid bars (edges) of lengths \( l_i \) consecutively joined at the vertices by revolving joints. It lies in the plane, its vertices may move, and the edges may freely rotate around endpoints and intersect each other. This makes various planar configurations of the robot arm. In the engineering context one can think of this as a closed chain with one link being “telescopic”.

Let us make this precise. A configuration of a robot arm is defined as an \((n+1)\)-tuple of points \( R = (r_0, \ldots, r_n) \) in the Euclidean plane \( \mathbb{R}^2 \) such that \( |r_i - r_{i+1}| = l_i, i = 1, \ldots, n \). Each configuration carries a natural orientation given by vertices’ order. To factor out the action of orientation-preserving isometries of the plane \( \mathbb{R}^2 \), we consider only configurations with two first vertices fixed: \( r_0 = (0, 0), r_1 = (l_1, 0) \). The set of all such planar configurations of a robot arm is called the moduli space of a robot arm. We denote it by \( M^0(L) \). It is a subset of Euclidean space \( \mathbb{R}^{2n-2} \) and inherits its topology and differentiable space structure so that one can speak of smooth mappings and diffeomorphisms in this context. After these preparations it is obvious that the moduli space of any planar robot arm is diffeomorphic to the torus \((S^1)^{n-1}\). We will use its parametrization by angle-coordinates \( \beta_i \) (that is, by angles between \( r_0r_1 \) and \( r_kr_{k+1}, k = 1, \ldots, n - 1 \)).

In this paper we consider the oriented (signed) area as a function on \( M^0(L) \).

**Definition 2.1.**

For any configuration \( R \) of \( L \) with vertices \( r_i = (x_i, y_i), i = 0, \ldots, n \), its (doubled) oriented area \( A(R) \) is defined by

\[
2A(R) = (x_0y_1 - x_1y_0) + \cdots + (x_0y_n - x_ny_0).
\]

In other words, we add the connecting side \( r_0r_0 \) turning a given configuration \( R \) into an \((n+1)\)-gon and compute the oriented area of the latter. Obviously, \( A(R) \) is a smooth function on the moduli space \( M^0(L) \) of any robot arm \( L \).

3. Critical configurations. 3-arms

A configuration \( R = (r_0, \ldots, r_n) \) of a robot arm \( L = (l_1, \ldots, l_n) \) is cyclic if all its vertices lie on a circle. A configuration is quasicyclic (a QC-configuration for short) if all its vertices lie either on a circle or on a (straight) line. A configuration is closed cyclic if the last and the first vertices coincide: \( r_0 = r_n \). A configuration is diacyclic if it is cyclic and the “connecting side” \( r_0r_0 \) is a diameter of the circumscribed circle (“diacyclic” is a sort of shorthand for “diametrically cyclic”). In other words, the connecting side \( r_0r_0 \) passes through the center of the circumscribed circle or, equivalently, each interval \( r_0r_k \) is orthogonal to the interval \( r_kr_n \) for \( k = 1, \ldots, n - 1 \).