Unramified Brauer group of the moduli spaces of $\text{PGL}_r(\mathbb{C})$-bundles over curves

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Abstract: Let $X$ be an irreducible smooth complex projective curve of genus $g$, with $g \geq 2$. Let $N$ be a connected component of the moduli space of semistable principal $\text{PGL}_r(\mathbb{C})$-bundles over $X$; it is a normal unirational complex projective variety. We prove that the Brauer group of a desingularization of $N$ is trivial.

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1. Introduction

Let $X$ be an irreducible smooth complex projective curve, with genus $g(X) = g \geq 2$. For a fixed algebraic line bundle $\mathcal{L}$ over $X$, let $N_X(r, \mathcal{L})$ be the coarse moduli space of semistable vector bundles over $X$ of rank $r$ and determinant $\mathcal{L}$. It is a normal unirational complex projective variety, and if degree($\mathcal{L}$) is coprime to $r$, then $N_X(r, \mathcal{L})$ is known to be rational [9, 14]. Apart from this coprime case, and the single case of $g = r = \text{degree}(\mathcal{L}) = 2$ when $N_X(r, \mathcal{L}) = \mathbb{P}^2_{\mathbb{C}}$ (see [12, Theorem 2]), the rationality of $N_X(r, \mathcal{L})$ is an open question in every other case. See [7] for rationality of some other types of moduli spaces associated to $X$. In [4] we showed that the Brauer group of a desingularization of $N_X(r, \mathcal{L})$ vanishes. We recall that the Brauer group of a smooth projective variety is a birational invariant, and its vanishing is a necessary condition for the variety to be rational.

Here we consider the coarse moduli space $N_X(r, d)$ of semistable principal $\text{PGL}_r(\mathbb{C})$-bundles of topological type $d \in \{0, \ldots, r - 1\}$ over $X$. We recall that a principal $\text{PGL}_r(\mathbb{C})$-bundle $P/X$ is said to be of topological type $d$ if

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the associated $\mathbb{P}^{r-1}$-bundle is isomorphic to $\text{Proj}(\mathcal{E})$ for some rank $r$ vector bundle $E$ whose degree is congruent to $d$ modulo $r$. This $N_{r}(r, d)$ is an irreducible normal unirational complex projective variety. We prove that the Brauer group of a desingularization of $N_{r}(r, d)$ is zero (see Theorem 4.2).

When $g = 2$, the moduli space $N_{r}(2, 0)$ is a quotient of $\mathbb{P}^{3}_{C}$ by a faithful action of the abelian group $(\mathbb{Z}/2\mathbb{Z})^{4}$. Using [1, Proposition 9.1] it follows that there is an extension of $(\mathbb{Z}/2\mathbb{Z})^{4}$ by $(\mathbb{Z}/2\mathbb{Z})$ whose induced action on the projective space $\mathbb{P}^{3}_{C}$ is linearized by $\mathcal{O}_{\mathbb{P}^{3}_{C}}(1)$. Hence in this special case the quotient $N_{r}(2, 0)$ is rational by [8, Theorem 1.4].

2. Preliminaries

We continue with the above set-up and notation. Let $N_{r}(r, d)$ denote the coarse moduli space of $S$-equivalence classes of all semistable principal $\text{PGL}_{r}(\mathbb{C})$-bundles of topological type $d$ over $X$. For notational convenience, $N_{r}(r, d)$ will also be denoted by $N$. See [10, Section 5], [3, 13] for $N$.

Let $M_{r}(r, \mathcal{L}_{X})$ denote the coarse moduli space of $S$-equivalence classes of semistable vector bundles over $X$ of rank $r$ and determinant $\mathcal{L}_{X}$. Let $\Gamma$ be the group of all isomorphism classes of algebraic line bundles $\tau$ over $X$ such that $\tau^{\otimes r} = O_{X}$. This group $\Gamma$ has the following natural action on $M_{r}(r, \mathcal{L}_{X})$: the action of any $\tau \in \Gamma$ sends any $E \in M_{r}(r, \mathcal{L}_{X})$ to $E \otimes \tau$. The moduli space $N$ is identified with the quotient variety $M_{r}(r, \mathcal{L}_{X})/\Gamma$. Let

$$f: M_{r}(r, \mathcal{L}_{X}) \to M_{r}(r, \mathcal{L}_{X})/\Gamma = N$$

be the quotient morphism. For notational convenience, the moduli space $M_{r}(r, \mathcal{L}_{X})$ will also be denoted by $M_{\mathcal{L}_{X}}$. Let

$$M_{\mathcal{L}_{X}}^{st} \subset M_{\mathcal{L}_{X}}, \quad N^{st} \subset N$$

be the loci of stable bundles. The above action of $\Gamma$ on $M_{\mathcal{L}_{X}}$ preserves $M_{\mathcal{L}_{X}}^{st}$, and we have $f(M_{\mathcal{L}_{X}}^{st}) = N^{st}$.

3. The action of $\Gamma$

Consider the action of $\Gamma$ on $M_{\mathcal{L}_{X}}$ defined in Section 2. For any primitive $\tau \in \Gamma$, i.e., an element of $\Gamma$ of order $r$, let

$$M_{\mathcal{L}_{X}}^{\tau} = \{E \in M_{\mathcal{L}_{X}}: E \otimes \tau = E \} \subset M_{\mathcal{L}_{X}}$$

be the fixed point locus. Take any nontrivial line bundle $\tau \in \Gamma$ of order $r$. Let $\phi: Y \to X$ be the étale cyclic covering of degree $r$ given by $\tau$. We recall the construction of $Y$. Fix an isomorphism $\theta: \tau^{\otimes r} \to O_{X}$. Then $Y \subset \tau$ is the locus of all elements $v$ such that $\theta(v^{\otimes r})$ projects to $1$ by the natural projection $X \times \mathbb{C} \to \mathbb{C}$. Let $\beta: Y \to Y$ be a nontrivial generator of the Galois group $\text{Gal}(\phi) = \mathbb{Z}/r\mathbb{Z}$ of the covering $\phi$. The homomorphism $\xi \mapsto \beta^{*} \xi$ defines an action of $\text{Gal}(\phi)$ on $\text{Pic}^{d}(Y)$ for any $d \in \mathbb{Z}$. Let

$$\phi^{*}: \text{Pic}^{d}(X) \to \text{Pic}^{d}(Y)$$

be the pullback homomorphism $L \mapsto \phi^{*}L$. Let $K$ denote the kernel of $\phi^{*}$. This $K$ is a group of order $r$ generated by $\tau$. Let

$$\text{Nm}: \text{Pic}^{d}(Y) \to \text{Pic}^{d}(X), \quad \text{N}: \text{Pic}^{d}(Y) \to \text{Pic}^{d}(X)$$

be the norm homomorphism and the twisted norm morphism. We recall that $\text{Nm}$ takes a line bundle $\xi$ to the descent of $\otimes(\beta^{*}\xi)$ and $\text{N}$ sends a line bundle $\xi$ to $\text{Nm}(\xi) \otimes \tau^{\otimes (r-1)/2}$. For any $\sigma \in \Gamma$ consider the automorphism of $\text{Pic}^{d}(Y)$ defined by $\xi \mapsto \xi \otimes \phi^{*}\sigma$. This defines an action of the group $\Gamma$ on $\text{Pic}^{d}(Y)$. The homomorphism $\phi^{*}$ in (4) is, clearly, $\Gamma$-equivariant. Also, the above defined kernel $K$ acts trivially on $\text{Pic}^{d}(Y)$. The morphism $\text{N}$ in (5) factors through the quotient morphism $\text{Pic}^{d}(Y) \to \text{Pic}^{d}(Y)/\Gamma$. The action of $\Gamma$ on $\text{Pic}^{d}(Y)$, clearly, commutes with the action of $\text{Gal}(\phi)$ defined earlier. As before, $\mathcal{L}_{X}$ is a line bundle on $X$. Let $\mathcal{L}_{X}$ be the open subscheme of $N^{-1}(\mathcal{L}_{X})$ where the action of $\text{Gal}(\phi)$ is fixed point free. It is a $\Gamma$-invariant open subscheme. Now we state a well-known result (cf. [13, Proposition 3.3]).