Hopf hypersurfaces in complex two-plane Grassmannians with generalized Tanaka-Webster $\mathcal{D}^\perp$-parallel structure Jacobi operator

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Received 11 January 2013; accepted 5 April 2013

Abstract: Regarding the generalized Tanaka-Webster connection, we considered a new notion of $\mathcal{D}^\perp$-parallel structure Jacobi operator for a real hypersurface in a complex two-plane Grassmannian $G_2(C^{m+2})$ and proved that a real hypersurface in $G_2(C^{m+2})$ with generalized Tanaka-Webster $\mathcal{D}^\perp$-parallel structure Jacobi operator is locally congruent to an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{HP}^n$ in $G_2(C^{m+2})$, where $m = 2n$.

MSC: 53C40, 53C15

Keywords: Complex two-plane Grassmannian • Hopf hypersurface • Generalized Tanaka-Webster connection • Structure Jacobi operator

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1. Introduction

Regarding real hypersurfaces with parallel curvature tensor, many differential geometers were studied either in complex projective spaces or in quaternionic projective spaces ([7, 11, 12]). From another perspective, it is interesting to classify real hypersurfaces in complex two-plane Grassmannians with parallel shape operator, structure Jacobi operator and Ricci tensor (See [5, 6, 13–18]).

As an ambient space, a complex two-plane Grassmannian $G_2(C^{m+2})$ consists of all complex two-dimensional linear subspaces in $C^{m+2}$. This Riemannian symmetric space is the unique compact irreducible Riemannian manifold being

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equipped with both the Kähler structure $J$ and the quaternionic Kähler structure $\mathcal{J}$ not containing $J$. Then, we could naturally consider two geometric conditions for hypersurfaces $M$ in $G_2(C^{m+2})$, namely, that the 1-dimensional distribution $\{\xi\} = \text{Span}\{\xi\}$ and the 3-dimensional distribution $\mathcal{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator $A$ of $M$ ([3]), where the Reeb vector field $\xi$ is defined by $\xi = -JN$. $N$ denotes a local unit normal vector field of $M$ in $G_2(C^{m+2})$ and the almost contact 3-structure vector fields $\xi_\nu$ are defined by $\xi_\nu = -J_\nu N$ ($\nu = 1, 2, 3$). By using the result in Alekseevskii [1], Berndt and Suh [3] proved the following:

**Theorem A.**

Let $M$ be a connected orientable real hypersurface in $G_2(C^{m+2})$, $m \geq 3$. Then both $\{\xi\}$ and $\mathcal{D}^\perp$ are invariant under the shape operator of $M$ if and only if

(A) $M$ is an open part of a tube around a totally geodesic $G_2(C^{m+1})$ in $G_2(C^{m+2})$, or

(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(C^{m+2})$.

When we consider the Reeb vector field $\xi$ in the expression of the curvature tensor $R$ for a real hypersurface $M$ in $G_2(C^{m+2})$, the structure Jacobi operator $R_\xi$ can be defined in such as

$$R_\xi(X) = R(X, \xi)\xi,$$

for any tangent vector field $X$ on $M$.

By using the structure Jacobi operator $R_\xi$, Jeong, Pérez and Suh considered a notion of parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for any vector field $X$ on $M$ and gave a non-existence theorem (See [5]).

On the other hand, the Reeb vector field $\xi$ is said to be Hopf if it is invariant under the shape operator $A$. The one-dimensional foliation of $M$ by the integral manifolds of the Reeb vector field $\xi$ is said to be the Hopf foliation of $M$.

We say that $M$ is a Hopf hypersurface in $G_2(C^{m+2})$ if and only if the Hopf foliation of $M$ is totally geodesic. Using the formulas in Section 2 it can be easily checked that $M$ is Hopf if and only if the Reeb vector field $\xi$ is Hopf.

Moreover, the authors [6] considered the general notion of $\mathcal{D}^\perp$-parallel structure Jacobi operator defined by $\nabla_\xi R_\xi = 0$, $\nu = 1, 2, 3$, which is weaker than the notion of the parallel structure Jacobi operator mentioned above. They gave a non-existence theorem as follows:

**Theorem B.**

There do not exist any connected Hopf real hypersurfaces in $G_2(C^{m+2})$, $m \geq 3$, with $\mathcal{D}^\perp$-parallel structure Jacobi operator if the principal curvature $\alpha$ is constant along the direction of $\xi$.

Now, instead of Levi-Civita connection for real hypersurfaces in Kähler manifolds, we consider another new connection named generalized Tanaka-Webster connection (in short, the $g$-Tanaka-Webster connection) $\tilde{\nabla}^{(k)}$ for a non-zero real number $k$ (See [8]). This new connection $\tilde{\nabla}^{(k)}$ can be regarded as a natural extension of Tanno’s generalized Tanaka-Webster connection $\tilde{\nabla}$ for contact metric manifolds. Actually, Tanno [20] introduced the generalized Tanaka-Webster connection $\tilde{\nabla}$ for contact Riemannian manifolds by using the canonical connection on a nondegenerate, integrable $CR$ manifold.

On the other hand, the original Tanaka-Webster connection ([19, 21]) was given as a unique affine connection on a non-degenerate, pseudo-Hermitian $CR$ manifold associated with the almost contact structure. In particular, if a real hypersurface in a Kähler manifold satisfies $\phi A + A \phi = 2k \phi$ ($k \neq 0$), then the $g$-Tanaka-Webster connection $\tilde{\nabla}^{(k)} = \phi A + A \phi$ coincides with the Tanaka-Webster connection.

In [10], using this $g$-Tanaka-Webster connection $\tilde{\nabla}^{(k)}$, we considered the notion of Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection, that is, $\tilde{\nabla}^{(k)}_\xi R_\xi = 0$. We gave a non-existence theorem as follows:

**Theorem C.**

There does not exist any Hopf hypersurface in a complex two-plane Grassmannian $G_2(C^{m+2})$, $m \geq 3$, with Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection.