A concise quantum mechanical treatment of the forced damped harmonic oscillator

Ti Jun Li

Department of Physics, Heze University, Shandong 274015, PR China

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Abstract: By selecting a right generalized coordinate \( X \), which contains the general solutions of the classical motion equation of a forced damped harmonic oscillator, we obtain a simple Hamiltonian which does not contain time for the oscillator such that the Schrödinger equation and its solutions can be directly written out in \( X \) representation. The wave functions in \( x \) representation are also given with the help of the eigenfunctions of the operator \( \hat{X} \) in \( x \) representation. The evolution of \( \langle \hat{X} \rangle \) is the same as in the classical mechanics, and the uncertainty in position is independent of an external influence; one part of energy mean is quantized and attenuated, and the other is equal to the classical energy.

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1. Introduction

The quantum model for a damped harmonic oscillator is applied in many fields, for example, in quantum theory of fields, quantum optics, solid theory and the quantum theory of mesoscopic circuits. Several techniques, such as the invariant operator method, the propagator method, the unitary transformation method and so on, are used for the dissipative systems [1–7].

In this paper, a very simple method is introduced for studying the system. First we get a standard Hamiltonian for the system by selecting a right generalized coordinate \( X \), and write out the Schrödinger equation and the corresponding wave functions \( \psi_{\alpha}(X, t) \) in \( X \) representation. Afterwards we give the wave functions \( \psi_{\alpha}(x, t) \) with the help of the eigenfunctions of the operator \( \hat{X} \) in \( x \) representation. Finally the expectation value \( \langle \hat{X} \rangle \), \( \langle \hat{E} \rangle \), and the uncertainty in position are discussed.

2. The Hamiltonian

We start from the equation of motion for a damped driven harmonic oscillator:

\[
 m\ddot{x} + c\dot{x} + kx = f(t),
\]

where \( m, c, k \), and \( f(t) \) are the mass of the harmonic oscillator, damping constant, elasticity coefficient, and time-dependent driving force, respectively.
Since we only deal with underdamped oscillators \((c^2 < 4mk)\), the classical solution corresponding to Eq. (1) is

\[
x = \frac{1}{m\omega} \exp \left( -\frac{ct}{2m} \right) \Re \left\{ i \exp(-i\omega t) \int_0^t \exp \left( \frac{ct}{2m} \right) t(\tau) \exp(i\omega \tau) d\tau + \sqrt{2m\omega \hbar} \ x(0) \exp(-i\omega t) \right\} \equiv \hat{x},
\]

where

\[
\hat{k} = k - \frac{c^2}{4m}, \quad (3)
\]

\[
\omega_0 = \sqrt{\frac{k}{m}}, \quad (4)
\]

\[
\omega = \sqrt{\frac{k}{m}} = \sqrt{\omega_0^2 - \frac{c^2}{4m^2}}, \quad (5)
\]

\(x(0)\) is a complex number. In order to give a simple Hamiltonian, we select the generalized coordinate:

\[
X = \exp \left( \frac{ct}{2m} \right) (x - \bar{x}), \quad (6)
\]

the Lagrangian corresponding to Eq. (1) is

\[
L = \frac{1}{2} m \dot{X}^2 - \frac{1}{2} \hat{k} X^2, \quad (7)
\]

so the generalized momentum is

\[
P = \frac{\partial L}{\partial \dot{X}} = m \dot{X}, \quad (8)
\]

the Hamiltonian is introduced:

\[
H = (-L + P \dot{X})_{(\dot{X} = \dot{\bar{X}})} = \frac{P^2}{2m} + \frac{1}{2} \hat{k} X^2, \quad (9)
\]

so the corresponding Schrödinger equation is

\[
\hat{H} \psi(X, t) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial X^2} + \frac{1}{2} \hat{k} X^2 \right) \psi(X, t), \quad (11)
\]

and its solutions are

\[
\psi_n(X, t) = N_n \psi_0(X \alpha) \exp \left( -\frac{\alpha^2 X^2}{2} \right) \exp \left( -\frac{iE_n t}{\hbar} \right), \quad (12)
\]

where

\[
\alpha = \sqrt{\frac{m \omega}{\hbar}}, \quad (13)
\]

\[
E_n = \left( n + \frac{1}{2} \right) \hbar \omega, \quad (14)
\]

\[
N_n = \sqrt{\frac{\alpha}{\sqrt{\pi} 2^n n!}}, \quad n = 0, 1, 2, \ldots. \quad (15)
\]

### 3. Wave functions in \(X\) representation

In order to obtain the wave functions \(\psi_n(x, t)\) in \(X\) representation, the eigenvectors of the operator \(\hat{X}\) in \(X\) representation need to be given. Therefore we suppose that the eigen equation of \(\hat{X}\) is

\[
\hat{X} |X'\rangle = X' |X'\rangle, \quad (16)
\]

where \(|X'\rangle\) stands for the eigenvector of \(\hat{X}\) with eigenvalue \(X'\). According to Eq. (6) we have

\[
\hat{X} = \exp \left( \frac{ct}{2m} \right) (\hat{x} - \bar{x}), \quad (17)
\]

where \(\hat{x}\) is the coordinate operator. Substituting Eq. (17) into (16), we have

\[
\hat{X} |X'\rangle = \left( \hat{x} + X' \exp \left( -\frac{ct}{2m} \right) \right) |X'\rangle. \quad (18)
\]

### 3.2. Wave functions in \(x\) representation

In order to list the wave functions given here:.

\[
H = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \hat{k} X^2 \right) \psi(X, t), \quad (11)
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