The relative motion of membranes

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Abstract: The relative classical motion of membranes is governed by the equation $w^{a\beta\gamma\delta} = R^{a\beta\gamma\delta}_{\beta\alpha\delta}$, where $w$ is the hessian. This is a generalization of the geodesic deviation equation and can be derived from the lagrangian $p \cdot \dot{r}$. Quantum mechanically the picture is less clear. Some quantizations of the classical equations are attempted so that the question as to whether the Universe started with a quantum fluctuation can be addressed.

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1. Introduction

The geodesic deviation equation can be thought of as a way of expressing the Ricci identity. It can be much generalized to equations describing relative motion of many sorts of system; the easiest way of doing this is by the lagrange method, using a lagrangian of the form $p \cdot \dot{r}$, where $p$ is the momentum of a given system and $r$ is a separation. Here attention is restricted to the given system being a membrane so that questions in contemporary membrane cosmology [6] can be approached. The geodesic equations are sometimes thought of as a consequence of field equations [2]. So far the geodesic deviation equations appear to be independent of any set of field equations, and the same is true of its generalizations. This could be thought of as a good thing, as it means that any results involving them are independent of specific field equations, or as a bad thing as it does not allow choices to be made as to which field equations are best.

In Sec. 2 the general lagrangian theory allowing relative motion equations to be derived is presented. The main feature of this theory is that the variables are cross-conjugate rather than self-conjugate: this mean that the momentum associated with a given system $p$ is conjugate to the separation $r$ and that the momentum associated with the extension $P$ is conjugate to $x$. This causes all sorts of technical problems, because it does not allow the separation part to be added on linearly and approximated. The extended theory is a gauge theory, as has been described for the point particle [9]. Two systems in particular are used to illustrate cross-conjugate lagrangians, the point particle and the membrane, strings having been looked at previously [10]. In Sec. 3 the resulting relative motion equations are calculated in the case of maximal symmetry. Here the relative motion of membranes is reduced right down so that it is similar to that of point particles; however membranes can have their own dynamical degrees of freedom [3]. In Sec. 4 attempts are made to quantize the theory. There seems to be five approaches to this.
The first is if one thinks as the relative motion equations as being embedded in some general set of field equations, quantize these field equations, this is not looked at here. The second is that the relative motion lagrangian is an example of a gauge theory. This allows it to be quantized by formal methods, this has been done for the point particle [9]. The problem with this is that, although the Klein–Gordon equation can be recovered, the resulting wavefunctions do not seem to be separable into $x$ and $r$ parts; also it assumes that one knows how to quantize the given system. This approach is not looked at because of lack of wavefunction separability. The third is to assume that methods used to quantize the extended system can be applied to the extended system, this has been done for the point particle [1]. Here a naive quantization of the membrane is applied to the extended system. The fourth is to quantize membrane fluctuations [4]. The fifth is to apply assorted substitutions which give back plausible looking classical equations, this is done here to try to quantize the extended part of the system. Sec. 5 is the conclusion. Things not looked at include: firstly, any derivation of relative motion equations by second variation of standard lagrangians, this has been done for strings [10], secondly, anything to do with spin, fermions, torsion, vector fields, or fluids, geodesic deviation has been generalized to include spin [7], thirdly, any study of gravitational waves [8], fourthly, any detailed investigation of the relationship between relative motion equations and field equations, finally, any application of the equations to specific configurations except maximal symmetry. For a lagrangian theory $\mathcal{L}$, with action $I$, the momentum and hessian are:

$$p^{\alpha} \equiv \frac{\delta I}{\delta q^{\alpha}} = \frac{\partial \mathcal{L}}{\partial \dot{q}^{\alpha}}, \quad \omega^{\alpha \beta} \equiv \frac{\delta^{2} I}{\delta q^{\alpha} \delta q^{\beta}} = \frac{\partial^{2} \mathcal{L}}{\partial \dot{q}^{\alpha} \partial \dot{q}^{\beta}}, \quad (1)$$

where $q = x$ if the lagrangian is that of an extended object. Small letters $p, \omega$ are used for the given system, and capitals $\mathcal{P}, \mathcal{W}$ for the system extended to a cross-conjugate deviating system. Roman $p$ used for the p-brane. Greek letters are used for spacetime indices, latin letters for internal indices. Analogies and generalizations of the geodesic deviation equation are here called relative motion equations, rather than deviation equations. Lower case "lagrangian" and "hessian" throughout. All other notation is as in Hawking and Ellis (1973) [5].

2. Cross-conjugate actions

The lagrangian is generalized from that of a given system $\mathcal{L}$ to the relative motion lagrangian $\mathcal{L}_{r}$, the action is taken as

$$S = \int_{t_1}^{t_2} dt \mathcal{L} \quad \mathcal{L}_{r} = a_{1} \mathcal{L} + a_{2} \dot{r} \cdot p. \quad (2)$$

$r$ is a separation vector, $a_{1}$ and $a_{2}$ and constants, typically $a_{1} = 0$ as the equation of motion $\dot{p} = 0$ follows from the second part of (2), and $a_{2} = 1$. The lagrangian $\mathcal{L}$ has momentum from which a cross-product has to be formed with $\dot{r}$, typically this means that the momentum has one spacetime index so that $\dot{r} \cdot p = i^{\alpha} p_{\alpha}$. The integral in (2) takes that form for the point particle, however for membranes it is over all the internal indices, and for fields it is usually over some universal time $t$; similarly the dot on $r$ is proper time for the point particle, all the internal indices for a membrane, i.e. $\dot{r} \rightarrow \partial_{\tau} r$, and for fields it usually represents differentiation with respect to some universal time.

For $\mathcal{L}_{r} = \mathcal{L}_{r}(i, \dot{r})$ the Ricci identity is

$$\Delta i^{\alpha} = \frac{D}{d \tau} \Delta i^{\alpha} + R_{\beta \gamma \delta}^{\alpha} \delta x^{\gamma} \delta x^{\delta}. \quad (3)$$

Using (3) $\delta x$ and $\Delta r$ variations of (2) give

$$- \frac{D}{d \tau} \frac{\partial \mathcal{L}_{r}}{\partial \delta x^{\alpha}} + R_{\beta \gamma \delta}^{\alpha} \frac{\partial \mathcal{L}_{r}}{\partial \delta r^{\beta}} = 0, \quad - \frac{D}{d \tau} \frac{\partial \mathcal{L}_{r}}{\partial \dot{r}^{\alpha}} = 0, \quad (4)$$

respectively. For the specific form of the Lagrangian in (2), substituting (1) and (2) into (4) gives the momenta and hessian

$$\mathcal{P}_{r}^{\alpha} = \frac{\partial \mathcal{L}}{\partial \dot{r}^{\alpha}} = a_{2} \frac{\partial \mathcal{L}}{\partial x^{\alpha}} = a_{2} \dot{r}^{\alpha},$$

$$\mathcal{P}_{x}^{\alpha} = \frac{\partial \mathcal{L}}{\partial \dot{x}^{\alpha}} = a_{1} \frac{\partial \mathcal{L}}{\partial x^{\alpha}} + a_{2} \frac{\partial^{2} \mathcal{L}}{\partial x^{\gamma} \partial x^{\delta} \partial x^{\delta}} = a_{1} \dot{x}^{\alpha} + a_{2} \dot{w}_{\mu}^{\alpha},$$

$$\mathcal{W}_{\mu}^{\alpha} = 0,$$

$$\mathcal{W}_{\mu}^{\alpha} = \mathcal{W}_{\mu}^{\alpha} = a_{2} \dot{w}_{\mu}^{\alpha},$$

$$\mathcal{W}_{\mu}^{\alpha} = a_{1} \dot{w}_{\mu}^{\alpha} + a_{2} \dot{x}^{\gamma} \frac{\partial^{2} \mathcal{L}}{\partial x^{\gamma} \partial \dot{x}^{\delta} \partial \dot{x}^{\delta}}, \quad (5)$$

which allows the general form of the relative motion Eqs. (4) to be expressed in term of the momentum

$$\dot{p}^{\alpha} = 0, \quad \dot{P}_{x}^{\alpha} = a_{2} R_{\gamma \delta}^{\alpha} \delta x^{\gamma} \delta x^{\delta} \dot{r}^{\beta}, \quad (6)$$

where $P_{x}^{\alpha}$ is given by (5) and so has the hessian of the original system in it. In general metric variation of (2) gives stresses which do not have an immediate interpretation.

For the point particle the lagrangian can be taken as

$$\mathcal{L} = - m \ell, \quad \ell \equiv \sqrt{- \dot{x}^{2}}. \quad (7)$$