On the feasibility of charged wormholes

Peter K.F. Kuhffittig

Department of Mathematics, Milwaukee School of Engineering, Milwaukee, Wisconsin 53202-3109 USA

Abstract: While wormhole spacetimes are predictions of the general theory of relativity, specific solutions may not be compatible with quantum field theory. This paper modifies the charged wormhole model of Kim and Lee with the aim of satisfying an extended version of a quantum inequality due to Ford and Roman. The modified metric may be viewed as a solution of the Einstein fields equations representing a charged wormhole that is compatible with quantum field theory.


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1. Introduction

Wormholes are handles or tunnels in the geometry of spacetime connecting two distinct regions of our Universe or of completely different universes. The pioneer work of Morris and Thorne [1] has shown that, being solutions of the Einstein field equations, macroscopic wormholes may be actual physical objects that could even be traversed by humanoid travelers. Unlike black holes, which are also predictions of Einstein’s theory, wormholes can only be held open by the use of “exotic” matter; such matter violates the weak energy condition.

Because of the close connection between space and time, general relativity is able to tolerate science-fiction type phenomena such as wormholes and even time travel, as exemplified by the Gödel solution. Quantum field theory, on the other hand, is not so forgiving: it places severe restrictions on the existence of traversable wormholes [2–5]. In fact, according to Ford and Roman [4, 5], the wormholes discussed in Ref. [1] could not exist on a macroscopic scale. Interesting exceptions are the wormholes discussed in Refs. [6] and [7], but they are subject to extreme fine-tuning. This fine-tuning became an issue in seeking compatibility with quantum field theory by a suitable extension of the quantum inequalities [8, 9]. Given that exotic matter is rather problematical, the idea behind the extension was to strike a balance between reducing the size of the exotic region and the concomitant fine-tuning of the metric coefficients. One can only be accomplished at the expense of the other.

A particularly interesting generalization of the Morris-Thorne wormhole can be obtained by the addition of an electric charge, as proposed by Kim and Lee [10, 11]. The resulting spacetime is a combination of a Morris-Thorne spherically symmetric static wormhole and a Reissner-Nördstrom spacetime.
As in the case of black holes, wormholes with an electric charge have been of interest for some time. For example, by adding an electric charge, Gonzales, Guzman, and Sarbach [12] studied the possibility of stabilizing a wormhole supported by a ghost scalar field, discussed in their earlier papers [13, 14].
Rotating and magnetized wormholes supported by phantom scalar fields are discussed in Ref. [15]. (A ghost scalar field is often considered a simple example of phantom energy, which is itself of interest in a wormhole setting since it leads to a violation of the weak energy condition.)

The aim of this paper is to show that a relatively small modification of the metric describing a charged wormhole suffices to satisfy an extended version of the Ford-Roman inequality, thereby making such a wormhole compatible with quantum field theory. The modified model is also a solution of the Einstein field equations.

2. Traversable wormholes
The spacetime geometry of a traversable wormhole can be described by the metric

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $\beta(r) \to 0$ and $\alpha(r) \to 0$ as $r \to \infty$ and where $\alpha(r)$ has a continuous derivative. (We are using units in which $c = G = 1$.) The function $\beta = \beta(r)$ is called the redshift function, which must be everywhere finite to prevent an event horizon. The function $\alpha = \alpha(r)$ is related to the shape function $b = b(r)$:

$$e^{2\alpha(r)} = \frac{1}{1 - \frac{b(r)}{r}}.$$ 

So $b(r) = r(1 - e^{-2\alpha(r)})$. (Observe that $b'(r)$ is continuous and that $\frac{b'}{r} \to 0$ as $r \to \infty$.) The minimum radius $r = r_0$ is called the throat of the wormhole, where $b(r_0) = r_0$. Also, $b'(r_0) \leq 1$, referred to as the flare-out condition in Ref. [1]. It follows that $\alpha$ has a vertical asymptote at $r = r_0$:

$$\lim_{r \to r_0^+} \alpha(r) = +\infty.$$ 

To hold a wormhole open, the weak energy condition (WEC) must be violated. The WEC states that the stress-energy tensor $T_{\mu\nu}$ must obey

$$T_{\mu\nu}u^\mu u^\nu \geq 0$$

for all time-like vectors and, by continuity, all null vectors.

3. The quantum inequalities
To make this paper reasonably self-contained, we need a brief discussion of the quantum inequalities due to Ford and Roman [5], slightly extended in [8, 9].
In a series of papers, Ford and Roman (see Ref. [5] and references therein) discuss a type of constraint on the violation of the weak energy condition by means of certain quantum inequalities which limit the magnitude and time duration of negative energy. These inequalities place severe restrictions on the dimensions of Morris-Thorne wormholes.
One of these quantum inequalities, applied to different situations, deals with an inertial Minkowski spacetime without boundaries. If $u^\mu$ is the observer’s four-velocity, that is, the tangent vector to a timelike geodesic, then $\langle T_{\mu\nu}u^\mu u^\nu \rangle$ is the expectation value of the local energy density in the observer’s frame of reference. It is shown in Ref. [5] that

$$\frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{(T_{\mu\nu}u^\mu u^\nu)dt}{\tau^2 + \tau_0^2} \geq -\frac{3}{32\pi^2\tau_0^3}.$$  (2)

Here $\tau$ is the observer’s proper time and $\tau_0$ the duration of the sampling time. More precisely, the energy density is sampled in a time interval of duration $\tau_0$ which is centered around an arbitrary point on the observer’s worldline so chosen that $\tau = 0$ at this point. (See Ref. [5] for details.)
In a wormhole setting, a more convenient form is inequality (7) below, as we will see. Applied to spherically symmetric traversable wormholes in Ref. [1], it was found that none were able to meet this condition. As a result, the throat sizes could only be slightly larger than Planck length. The inequality was subsequently extended in Refs. [8, 9] to cover an entire region around the throat. It was then shown that it is possible to strike a balance between the size of the exotic region and the amount of fine-tuning required to achieve this reduction.

Before discussing the extended quantum inequality, we need to introduce the following length scales, modeled after the length scales in Ref. [5], which were introduced in Ref. [8]:

$$r_\infty \equiv \min \left[ r, \left( \frac{b(r)}{b'(r)} \right)^{\frac{1}{2}} \right].$$

(3)

It is shown that if $R_{\max}$ is the magnitude of the maximum curvature, then

$$R_{\max} \leq \frac{1}{r_\infty}.$$ 

So the smallest radius of curvature $r_c$ is

$$r_c \approx \frac{1}{\sqrt{R_{\max}}} \geq r_\infty.$$