Dynamics of a backlash chain

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Abstract: This paper studies the dynamical properties of a system with distributed backlash and impact phenomena. This means that it is considered a chain of masses that interact with each other solely by means of backlash and impact phenomena. It is observed the emergence of non-linear phenomena resembling those encountered in the Fermi-Pasta-Ulam problem.

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1. Introduction

In 1953, at Los Alamos, Enrico Fermi, John Pasta, and Stan Ulam developed a pioneer study that is nowadays called the Fermi-Pasta-Ulam (FPU) problem. They simulated numerically a mechanical system composed of identical masses coupled by nonlinear springs, fixed at the extreme points, using the computer MANIAC-1 (Mathematical Analyzer Numerical Integrator And Computer) [1]. Instead of using a linear model for the springs (i.e., the Hooke law) they adopted a nonlinear term, either quadratic (denoted FPU - α), or cubic (denoted FPU - β). Computer simulations revealed a complex quasi-periodic dynamics, considerably different from what linear systems would suggest. They expected that the energy introduced into the first mode would drift to the other modes until reaching equipartition of energy. However, they verified that almost all energy was back to the first mode after some time and, moreover, a recurrence phenomenon, due the the occurrence of a kind of replicas of the initial state for longer periods of time [2–4]. The FPU experiment marked the beginning of computational physics and nonlinear science and triggered a huge volume of research during the last decades [5–23].

Vibration with impacts occurs in many areas of science and technology. In the context of mechanical engineering, backlash is due to clearance between adjacent movable parts as in gears. Its effect is visible when movement is reversed and contact is lost momentarily, being re-established later when mating components produce some form of impact [24–36]. This strong non-linearity is not yet fully understood due to the diversity of effects involved and the dynamical analysis and control of backlash is still an open issue.

Inspired by the FPU problem and the backlash nonlinearity, this paper embeds both concepts and investigates the dynamics of systems with chain of masses having backlash and impacts. Several relevant studies of lattices with
hard collisions have been proposed during the last years [37–42]. The anomalous thermal conductivity was investigated, but this topic remains an important open area of research.

Bearing these ideas in mind, the present paper is organized as follows. Section 2 introduces the FPU problem and the dynamical description of backlash. Section 3 formulates a new problem, namely a chain of masses interconnected by means of backlash. The system is simulated and the dynamics of the distributed backlash chain of masses is analysed. Finally, section 4 draws the main conclusions.

2. Fundamental concepts

This section presents the main fundamental concepts applied in the sequel. Sub-section 2.1 formulates the classical FPU problem and sub-section 2.2 describes the dynamics of impacts.

2.1. The FPU dynamical system

The system formulated by Fermi, Pasta and Ulam consists of a chain of $N$ masses interconnected by springs (Fig. 1).

$$\ddot{x}_n = (x_{n+1} - x_n) - (x_n - x_{n-1}) + K_s [(x_{n+1} - x_n)^p - (x_n - x_{n-1})^p]$$

where $K_s$ is a parameter that reflects the strength of the nonlinearity and $n$, $1 \leq n \leq N$, is the index associated with each mass. The fixed extremes are represented by $n = 0$ and $n = N + 1$. When $p = 1$ it yields a linear model and when $p = 2$ (or $p = 3$) we have the FPU - $\alpha$ (or FPU - $\beta$) problem.

For the string $k$-th mode the sum of the kinetic and potential energies is given by:

$$E_k = \frac{1}{2} \left( A_k^2 + \omega_k^2 A_k^2 \right)$$

where $A_k$ is related to the displacements by the expression:

$$A_k = \sqrt{\frac{2}{N+1}} \sum_{n=1}^{N} x_n \sin \left( \frac{n k \pi}{N+1} \right)$$

Figure 1. The FPU problem: a chain of $N$ masses interconnected by springs.

and frequencies:

$$\omega_k = 2 \sin \left[ \frac{k \pi}{2(N+1)} \right]$$

Figure 2 depicts a typical time evolution of $E_k(t)$, $k = \{1, \cdots, 6\}$, for the FPU-$\alpha$, $K_s = 0.25$, for a period of time of $10^4$ sec. During the simulations it is adopted a Runge-Kutta 4 numerical integration with time step $dt = 10^{-2}$ sec, $N = 32$ and the initial conditions formulated by Fermi, Pasta and Ulam, namely $x_n(0) = \sin \left( \frac{n \pi}{N+1} \right)$, $\dot{x}_n(0) = 0$. It is clearly visible the recurrence phenomenon.

2.2. Dynamic backlash

In this sub-section we consider the description of backlash by means of the impacts and the law of conservation of momentum. This approach gives the net change in velocity of each body and the energy exchange during collisions [43–47].

Let us consider the impact of two bodies along surfaces that are normal to the line connecting their centres of mass. In the so-called central impact the two bodies have velocity components only along this line and no rotational or sliding effects occur.

Figure 3 depicts a mechanical model consisting of two masses $M_1$ and $M_2$ with backlash $\Delta$. Collision between the masses $M_1$ and $M_2$ occurs when $x_1 = x_2 - \frac{1}{2}$ or $x_1 = x_2 + \frac{1}{2}$.

The velocities before the impact $\{\dot{x}_1, \dot{x}_2\}$ are related to the new values $\{\dot{x}_1', \dot{x}_2'\}$ by means of the empirical law:

$$\dot{x}_1' - \dot{x}_2' = -\varepsilon (x_1 - x_2), \ 0 \leq \varepsilon \leq 1.$$