A finite volume method for solving the two-sided time-space fractional advection-dispersion equation

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Abstract: We present a finite volume method to solve the time-space two-sided fractional advection-dispersion equation on a one-dimensional domain. The spatial discretisation employs fractionally-shifted Grünwald formulas to discretise the Riemann-Liouville fractional derivatives at control volume faces in terms of function values at the nodes. We demonstrate how the finite volume formulation provides a natural, convenient and accurate means of discretising this equation in conservative form, compared to using a conventional finite difference approach. Results of numerical experiments are presented to demonstrate the effectiveness of the approach.

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1. Introduction

In recent times, the use of fractional partial differential equations to model transport processes has become popular among scientists, engineers and mathematicians. This surge in popularity is due to the growing number of real-world applications whose dynamics have been found to be more ably described by fractional models than by traditional integer-order models.

Fractional derivatives are a natural tool for modelling non-Fickian diffusion or dispersion processes, that is, processes exhibiting anomalous diffusion. The theoretical grounding is well established – Metzler and Klafter [1] show how fractional diffusion equations arise from continuous time random walk models with diverging characteristic waiting times and jump length variances. Particle transport in heterogeneous porous media is a well-known application where anomalous diffusion is observed. Zhang et al. [2] review many of the fractional models in this area, and describe several experiments where anomalous diffusion has been observed in the field.

In this paper, we consider the following two-sided time-space fractional advection-dispersion equation with variable coefficients:

\[
\begin{align*}
D_t^\alpha C(x, t) + \frac{\partial}{\partial x} \left( V(x, t) C(x, t) \right) &= 0 \\
\frac{\partial}{\partial x} \left[ K(x, t) \left( \beta \frac{\partial^\alpha C}{\partial x^\alpha} - (1 - \beta) \frac{\partial^\alpha C}{\partial (\Delta x)^\alpha} \right) \right] + S(x, t)
\end{align*}
\]

(1)
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on the interval \([a, b]\), subject to homogeneous Dirichlet boundary conditions. In the context of particle transport, \(C(x, t)\) represents the concentration of particles at position \(x\) and time \(t\). The symbol \(D_\alpha^t C(x, t)\) represents the Caputo fractional derivative of order \(\alpha\).

**Definition 1 (Caputo fractional derivative on \([0, \infty]\) [3]).**

\[
D_\alpha^t C(x, t) = \begin{cases} 
\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial C(x, \eta)}{\partial \eta} (t-\eta)^{-\alpha} d\eta, & 0 < \alpha < 1, \\
\frac{\partial}{\partial t} C(x, t), & \alpha = 1.
\end{cases}
\]

The symbols \(\partial^\alpha C/\partial x^\alpha\) and \(\partial^\alpha C/\partial (-x)^\alpha\) represent the left and right Riemann-Liouville fractional derivatives of order \(\alpha\).

**Definition 2 (Riemann-Liouville fractional derivatives on \([a, b]\) [3]).**

\[
\frac{\partial^\alpha C(x, t)}{\partial x^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial \xi^n} \int_a^x \frac{C(\xi, t)}{(x-\xi)^{\alpha+1-n}} d\xi,
\]

(2)

\[
\frac{\partial^\alpha C(x, t)}{\partial (-x)^\alpha} = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial \xi^n} \int_x^b \frac{C(\xi, t)}{(x-\xi)^{\alpha+1-n}} d\xi,
\]

(3)

where \(n\) is the smallest integer greater than or equal to \(\alpha\).

In this paper we consider regimes where \(0 < \alpha \leq 1\) and \(0 < \alpha \leq 1\). The inclusion of the skewness \(\beta \in [0, 1]\) in equation (1) allows for modelling of regimes where the forward and backward jump probabilities are different [4].

The remaining components of equation (1) are the velocity \(V(x, t)\), the anomalous dispersion coefficient \(K(x, t)\) and the source term \(S(x, t)\).

The literature on numerical methods for solving fractional differential equations is not as well-developed as for integer-order equations. Finite difference methods have dominated the literature. The L1 scheme [5] and variations thereof have formed the basis for many finite difference discretisations of Caputo time fractional derivatives, such as the widely used method of Lin and Xu [6].

In space too, finite difference methods have proved very popular. The pioneering work of Meerschaert et al. [4, 7–9] laid the groundwork for many of the popular finite difference discretisations for space-fractional derivatives. These are based on the equivalence of Riemann-Liouville and Grünwald-Letnikov fractional derivatives (for sufficiently smooth functions). The Grünwald-Letnikov definition is preferred numerically because it lends itself naturally to discretisation in a finite difference sense. A crucial observation was the necessity of using shifted formulas to ensure stability of these numerical schemes [7].

Many of these methods have since been generalised and extended in various ways, leading to non-standard finite difference schemes [10–12], finite difference schemes for problems of variable fractional order [13–16] and “fast” finite difference schemes [17–21] to name a few.

Other numerical methods have received less attention in the literature to date. The literature on finite volume methods for fractional partial differential equations in particular is still in early stages of development. A finite volume method for the space fractional advection-dispersion equation was proposed by Zhang et al. [22, 23], who chose to discretise the Riemann-Liouville derivatives directly, rather than using the Grünwald-Letnikov definition.

Previously we have considered a finite volume method for the two-sided space-fractional advection-dispersion equation with constant coefficients [24], based on the Grünwald-Letnikov definition, where we proved the stability and convergence of the method. In this paper, we extend the method to solve the two-sided time-space fractional advection-dispersion equation with variable coefficients. Additionally we derive a new result concerning the relationship between the finite difference method and the finite volume method in the constant coefficient case. We also demonstrate the improvement in accuracy provided by the finite volume method over the finite difference method for a given test problem.

The remainder of the paper is organised as follows. In Section 2 we derive equation (1) from conservation principles. In Section 3 we derive the new finite volume method for (1) and also make comparisons between it and the finite difference method in the constant coefficient case. In Section 4 we illustrate the method’s performance on test problems. Finally, we draw our conclusions in Section 5.

### 2. Derivation

The derivation of the advection–dispersion equation, whether standard or fractional, begins with the law of mass conservation which, in conservative form, is

\[
\frac{\partial C}{\partial t} = - \frac{\partial Q}{\partial x} + S(x, t)
\]

(4)

where \(Q\) is the flux. The flux comprises two components:

\[
Q(x, t) = V(x, t)C(x, t) + q(x, t).
\]

(5)