Abstract

By applying the monotone iterative technique and the method of lower and upper solutions, this paper investigates the existence of extremal solutions for a class of nonlinear fractional differential equations, which involve the Riemann-Liouville fractional derivative $D^q x(t)$. A new comparison theorem is also build. At last, an example is given to illustrate our main results.

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Key Words and Phrases: monotone iterative technique, nonlinear fractional differential equation, comparison theorem, extremal solutions

1. Introduction and preliminaries

Despite of the fact that the fractional calculus is as old as the classical one it started to be recognized as a powerful tool for analyzing the dynamics of the complex or hypercomplex systems (for details see [12], [8], [13], [9], [7], [3]). The beauty of this type of calculus is that it captures the properties of the nonlocal dynamics better that the other existing classical methods and techniques. The range of applications of the fractional calculus is very large at this moment and it is expected to increase in the nearest future (see for example [14], [11], [8], [5], [2], [1] and the references therein).
Many open problems remain to be investigated in the area of nonlinear fractional differential equations and for this reason new methods and techniques should be invented and applied for systems from the real world applications.

Having all above mentioned in mind, in this paper we study the following nonlinear fractional differential equations

\[
\begin{aligned}
(D^q x(t))' &= f(t, x(t), D^q x(t)), \\
D^q x(0) &= x_0^*, \quad t^{1-q} x(t)|_{t=0} = x_0,
\end{aligned}
\]

(1.1)

where \( t \in J = [0, T] \) \( (T > 0) \), \( f \in C(J \times \mathbb{R} \times \mathbb{R}) \), \( x_0^*, x_0 \in \mathbb{R} \) and \( D^q \) is the Riemann-Liouville fractional derivative of \( x \), and \( q \) is such that \( 0 < q < 1 \).

**Definition 1.1.** The Riemann-Liouville (R-L) fractional derivative of order \( \delta > 0 \) for a function \( f(t) \) is defined by

\[
D^\delta f(t) = \frac{1}{\Gamma(n-\delta)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\delta-1} f(s) ds, \quad n = [\delta] + 1,
\]

provided the right hand side is defined pointwise on \((0, \infty)\) and \([\delta]\) denoted the integer part of the order \( \delta \).

**Definition 1.2.** The Riemann-Liouville (R-L) fractional integral of order \( \delta \) for a function \( f \) is defined as

\[
I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s) ds, \quad \delta > 0,
\]

provided that such integral exists.

The rest of the paper is organized as follows.

In Section 2 we list several lemmas and a new comparison principle, playing an important role in the proof of the main results. Further, we consider the existence of the extremal solutions for a class of first order nonlinear differential equation involving the R-L fractional integral operator. In Section 3 we formulate sufficient conditions which guarantee that problem (1.1) has extremal solutions. A one-sided Lipschitz condition is imposed. In Section 4 an example is given to illustrate our main results.

2. Results for a differential equation involving R-L fractional integral operator

In this section, we study the following initial value problem for first order differential equation involving R-L fractional integral operator