Chapter 6
Calculus on Walsh and Vilenkin Groups

György Gát, Rodolfo Toledo

6.1 Introduction

The usual concept of differentiation is not suitable for functions which are locally constant. So it cannot be used either in the study of Walsh-Fourier series as with trigonometric series. However, Gibbs [16], Butzer and Wagner [2] introduced the concept of dyadic derivative which satisfies some of the usual properties of the differentiation, but not all. In this regard, for all function $f$ on the interval $[0, 1]$ and positive integer $n$ set

$$d_n f(x) := \sum_{j=0}^{n-1} 2^{-j-1} (f(x) - f(x + 2^{-j-1})), \quad (6.1)$$

where $+$ is the dyadic addition of two numbers on the interval $[0, 1]$ (see [27]). We say that $f$ is dyadic differentiable at the point $x$ if the limit

$$f^{[1]}(x) := \lim_{n \to \infty} d_n f(x)$$

exists and it is finite. Then we call $f^{[1]}$ the dyadic derivative of $f$.

The dyadic derivative is related to the Fourier series of functions in $L^1([0, 1])$ with respect to the Walsh-Paley system. We construct this system as follows. Denote by $P$ the set of positive integers and $N$ the set of non-negative integers. By the dyadic expansion of a number $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ in the interval $[0, 1]$ (in case $x$ is of the

György Gát
College of Nyíregyháza, Inst. of Math. and Comp. Sci., Nyíregyháza, P.O.Box 166., H–4400, Hungary, e-mail: gatgy@nyf.hu

Rodolfo Toledo
College of Nyíregyháza, Inst. of Math. and Comp. Sci., Nyíregyháza, P.O.Box 166., H–4400, Hungary, e-mail: toledo@nyf.hu

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form \( \frac{1}{2^n} \), where \( k, m \in \mathbf{P} \), is chosen the expansion which terminates in zeros) we obtain the Rademacher functions as:

\[
r_n(x) := (-1)^{x_n} \quad (x \in [0, 1], \ n \in \mathbf{N}).
\]

Set \( n = \sum_{j=0}^{\infty} n_j 2^j \) the binary expansion of the non-negative integer \( n \). Then the \textit{Walsh-Paley system} is given by the product of Rademacher functions:

\[
\omega_n(x) := \prod_{k=0}^{\infty} r_{nk}(x) \quad (x \in [0, 1], \ n \in \mathbf{N}).
\]

\( \omega_n \) is an orthonormal system consisting of dyadically differentiable functions such that

\[
\omega_n^{[1]}(x) = n \omega_n(x)
\]

for all \( x \in [0, 1] \) and \( n \in \mathbf{N} \). Note that the above property is analogous to the formula \((e^{int})' = ine^{int}\) for the usual derivative with respect to the classical trigonometric system.

A Walsh series \( \sum_{k=0}^{\infty} a_k \omega_k \) is said to be term by term dyadically differentiable at the point \( x \in [0, 1] \) if

\[
f(t) := \sum_{k=0}^{\infty} a_k \omega_k(t)
\]

exists and it is finite for \( t = x \) and \( t = x + 2^{-l} \ (l \in \mathbf{P}) \) and also if the dyadic derivative of \( f \) exists at the point \( x \) such that

\[
f^{[1]}(x) = \sum_{k=0}^{\infty} ka_k \omega_k(x).
\]

\textbf{Theorem 6.1 (Schipp [25])} If the sequence \( \langle ka_k \rangle \) decays monotonically to zero, then the Walsh series \( \sum_{k=0}^{\infty} a_k \omega_k \) is term by term dyadically differentiable at the point \( x \in [0, 1], \ x \neq 2^{-j} \ (j \in \mathbf{P}) \).

An integrable function on the interval \([0, 1]\) is called strongly differentiable if there is an integrable function on \([0, 1]\) denoted by \( df \) such that

\[
\lim_{n \to \infty} \| df - d_n f \|_1 = 0.
\]

In this sense functions of the Walsh-Paley system are also strongly differentiable and

\[
d \omega_n(x) = n \omega_n(x) \quad (n \in \mathbf{N}) \quad (6.2)
\]

almost everywhere on \( x \in [0, 1] \). On the other hand, the \textit{Walsh-Fourier coefficients}

\[
\hat{f}(k) = \int_{0}^{1} f(x) \omega_k(x) \, dx \quad (k \in \mathbf{N})
\]

of a strongly differentiable function \( f \) have the following expected statement.