2. **Metrizable Compact Spaces**

Our theme is the connections between the topology of $X$ and the abstract properties of $C(X)$. In this chapter we consider a simple, natural category of compact spaces: the metrizable ones.

We show that for a compact Hausdorff space $X$, metrizability is reflected in separability of $C(X)$. We consider two special spaces, the Hilbert cube and the Cantor set. We show that they are generic in the following sense: The metrizable compact Hausdorff spaces are precisely the closed subspaces of the Hilbert cube, and the quotients of the Cantor set.

2.1 The most accessible topological spaces are the metrizable ones. We open this chapter by stating some standard results about metrizable compact spaces.

(1) We have already mentioned: *In a metrizable compact space, every sequence has a convergent subsequence.*

(2) *Every metrizable compact space is separable* (i.e. has a countable dense subset).

(3) *On a compact metric space every continuous function is uniformly continuous.* Here a comment is in order.

A function $f$ on a set $X$ is, by definition, uniformly continuous under a metric $d$ on $X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x, y \in X, \quad d(x, y) < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon.$$  

Uniform continuity is not a *topological* property: A function may be uniformly continuous under a metric $d_1$ and not under a metric $d_2$ that determines the same topology. However, this cannot happen if the topology is compact.

2.2 **Theorem** Let $(X_1, d_1), (X_2, d_2), \ldots$ be metric spaces and put $X := \prod_{n \in \mathbb{N}} X_n$. For $x \in X$ and $n \in \mathbb{N}$ let $x_n$ be the $n$-th coordinate of $x$, so that $x = (x_n)_{n \in \mathbb{N}}$. Define

$$d(x, y) = \sup_n \{d_n(x_n, y_n) \land n^{-1}\} \quad (x, y \in X).$$

Then $d$ is a metric on $X$. The topology it determines is the product topology, i.e., if $(x_a)_{a \in A}$ is a net in $X$ and $x \in X$, then

$$d(x_a, x) \stackrel{a}{\longrightarrow} 0 \quad \Leftrightarrow \quad d_n(x_a, x_n) \stackrel{a}{\longrightarrow} 0 \quad (n \in \mathbb{N}).$$

In particular, the coordinate maps $X \rightarrow X_n$ are continuous.

**Proof**

(I) First, a preparatory observation: If $s, t, u, c \in [0, \infty)$ and $u \leq s + t$ then

$$u \land c \leq s \land c + t \land c.$$  

(The right hand member is $s + t$ or it is at least $c$.)

(II) Consequently, if $x, y, z \in X$, then for every $n$

$$d_n(x_n, y_n) \land n^{-1} \leq d_n(x_n, y_n) \land n^{-1} + d_n(y_n, z_n) \land n^{-1} \leq d(x, y) + d(y, z).$$

By taking the supremum over $n$ one sees that $d$ satisfies the triangle inequality, and actually is a metric.

(III) Let $(x_a)_{a \in A}$ be a net in $X$ and let $x \in X$.

- If $d(x_a, x) \stackrel{a}{\longrightarrow} 0$: Take $n$ in $\mathbb{N}$; we prove $d_n(x_a, x_n) \stackrel{a}{\longrightarrow} 0.$
For sufficiently large \( \alpha \) we have
\[
n^{-1} > d(x, x) \geq d_n(x_{\alpha, n}, x_n) \land n^{-1};
\]
then \( d(x, x) \) cannot be strictly less than \( d_n(x_{\alpha, n}, x_n) \). Thus, \( d_n(x_{\alpha, n}, x_n) \leq d(x, x) \) if \( \alpha \) is large enough, and \( d_n(x_{\alpha, n}, x_n) \xrightarrow{\alpha} 0. \)

- If \( d_n(x_{\alpha, n}, x_n) \xrightarrow{\alpha} 0 \) for every \( n \): Let \( \varepsilon > 0 \). Choose \( N \in \mathbb{N} \) with \( N^{-1} < \varepsilon \). For sufficiently large \( \alpha \),
\[
d_n(x_{\alpha, n}, x_n) \leq \varepsilon \quad \text{for } n = 1, ..., N.
\]
For such \( \alpha \) it follows from the definition of \( d \) that \( d(x, x) \leq \varepsilon. \)

Theorem 2.2 is especially of interest in conjunction with Tychonoff’s Theorem 1.22:

2.3 Theorem A Cartesian product of countably many metrizable compact spaces is metrizable and compact.

2.4 Examples

(1) The metrizable compact space \([0,1]^\mathbb{N}\) is called the Hilbert cube. (The same name is sometimes given to \([-1,1]^\mathbb{N}\). Topologically, these spaces are indistinguishable.)

In Theorem 2.8 we prove that every metrizable compact space is homeomorphic with a subspace of \([0,1]^\mathbb{N}\).

(2) Another special metrizable compact space is \(\{0,1\}^\mathbb{N}\).

The elements of \(\{0,1\}^\mathbb{N}\) are the indicators of the subsets of \(\mathbb{N}\). With the natural metric on \(\{0,1\}\) (the distance between 0 and 1 being 1), Theorem 2.2 yields the metric \(d\) on \(\{0,1\}^\mathbb{N}\):
\[
d(\mathbbm{1}_A, \mathbbm{1}_B) = \sup_n |\mathbbm{1}_A(n) - \mathbbm{1}_B(n)| \land n^{-1}.
\]

The following is a different “incarnation” of \(\{0,1\}^\mathbb{N}\). Let \(\mathcal{P}(\mathbb{N})\) be the collection of all subsets of \(\mathbb{N}\). The formula \(A \mapsto \mathbbm{1}_A\) determines a bijection \(\mathcal{P}(\mathbb{N}) \rightarrow \{0,1\}^\mathbb{N}\). For \(A, B \subset \mathbb{N}\) we have \(|\mathbbm{1}_A - \mathbbm{1}_B| = \mathbbm{1}_{A \Delta B}\) where \(A \Delta B = (A \cup B) \setminus (A \cap B)\).

Consequently, we can make a metric \(\delta\) on \(\mathcal{P}(\mathbb{N})\) by
\[
\delta(A, B) := \sup_{n \notin A \Delta B} n^{-1}
\]
(the right hand member being 0 if \(A \Delta B = \emptyset\)); this metric renders \(\mathcal{P}(\mathbb{N})\) a compact space and the map \(A \mapsto \mathbbm{1}_A\) an isometry.

(3) For every bounded closed interval \([a, b]\) we consider two subintervals
\[
[a, b]_0 := \left[a, a + \frac{b-a}{3}\right] \quad \text{and} \quad [a, b]_1 := \left[b - \frac{b-a}{3}, b\right].
\]

\[
\text{For such } \alpha \text{ it follows from the definition of } d \text{ that } d(x, x) \leq \varepsilon. \]

Theorem 2.2 is especially of interest in conjunction with Tychonoff’s Theorem 1.22:

2.3 Theorem A Cartesian product of countably many metrizable compact spaces is metrizable and compact.

2.4 Examples

(1) The metrizable compact space \([0,1]^\mathbb{N}\) is called the Hilbert cube. (The same name is sometimes given to \([-1,1]^\mathbb{N}\). Topologically, these spaces are indistinguishable.)

In Theorem 2.8 we prove that every metrizable compact space is homeomorphic with a subspace of \([0,1]^\mathbb{N}\).

(2) Another special metrizable compact space is \(\{0,1\}^\mathbb{N}\).

The elements of \(\{0,1\}^\mathbb{N}\) are the indicators of the subsets of \(\mathbb{N}\). With the natural metric on \(\{0,1\}\) (the distance between 0 and 1 being 1), Theorem 2.2 yields the metric \(d\) on \(\{0,1\}^\mathbb{N}\):
\[
d(\mathbbm{1}_A, \mathbbm{1}_B) = \sup_n |\mathbbm{1}_A(n) - \mathbbm{1}_B(n)| \land n^{-1}.
\]

The following is a different “incarnation” of \(\{0,1\}^\mathbb{N}\). Let \(\mathcal{P}(\mathbb{N})\) be the collection of all subsets of \(\mathbb{N}\). The formula \(A \mapsto \mathbbm{1}_A\) determines a bijection \(\mathcal{P}(\mathbb{N}) \rightarrow \{0,1\}^\mathbb{N}\). For \(A, B \subset \mathbb{N}\) we have \(|\mathbbm{1}_A - \mathbbm{1}_B| = \mathbbm{1}_{A \Delta B}\) where \(A \Delta B = (A \cup B) \setminus (A \cap B)\).

Consequently, we can make a metric \(\delta\) on \(\mathcal{P}(\mathbb{N})\) by
\[
\delta(A, B) := \sup_{n \notin A \Delta B} n^{-1}
\]
(the right hand member being 0 if \(A \Delta B = \emptyset\)); this metric renders \(\mathcal{P}(\mathbb{N})\) a compact space and the map \(A \mapsto \mathbbm{1}_A\) an isometry.

(3) For every bounded closed interval \([a, b]\) we consider two subintervals
\[
[a, b]_0 := \left[a, a + \frac{b-a}{3}\right] \quad \text{and} \quad [a, b]_1 := \left[b - \frac{b-a}{3}, b\right].
\]