9. \( C(X) \) determines \( X \)

It is a banality to observe that, if \( X \) and \( Y \) are homeomorphic topological spaces, then \( C(X) \) and \( C(Y) \) are isomorphic as vector spaces, as ordered sets, as rings, etc. More precisely, a homeomorphism \( \tau: X \to Y \) induces a bijection \( f \mapsto f \circ \tau \) of \( C(Y) \) onto \( C(X) \) that preserves all kinds of abstract structure. (See Theorem 9.1.)

In this chapter we consider the problem of the converse: If \( X \) and \( Y \) are topological spaces and if \( C(X) \) and \( C(Y) \) are isomorphic as ordered sets, say, must \( X \) and \( Y \) be homeomorphic?

Put thus crudely, the question has, of course, a negative answer: \( C(X) \) and \( C(Y) \) are in every reasonable way isomorphic in case \( X \) and \( Y \) carry their trivial topologies (\( \{\emptyset, X\} \) and \( \{\emptyset, Y\} \)) since then the only continuous functions are the constants, whereas there may not be a bijection \( X \to Y \).

It will turn out, however, that we can obtain interesting results by restricting ourselves to realcompact spaces. (See Theorem 9.2.)

First, the easy part:

9.1 Theorem Let \( X \) and \( Y \) be topological spaces, \( \tau \) a homeomorphism \( Y \to X \). For \( f \) in \( C(X) \), put

\[
T(f) := f \circ \tau.
\]

Then \( T \) is a bijection \( C(X) \to C(Y) \). It is linear and multiplicative, i.e.

\[
T(fg) = T(f) \cdot T(g) \quad (f, g \in C(X)).
\]

\( T \) is an order isomorphism, i.e.

\[
T(f) \leq T(g) \iff f \leq g \quad (f, g \in C(X)).
\]

If \( X \) and \( Y \) are compact, then:

\[
\|T(f)\|_\infty = \|f\|_\infty \quad (f \in C(X)),
\]

so that \( T \) is an isometry with respect to the metrics induced by the sup-norm.

We leave the proof to the reader.

A synopsis of the results in the other direction is given in:

9.2 Theorem Let \( X \) and \( Y \) be realcompact topological spaces. Suppose there exists a linear bijection \( C(X) \to C(Y) \) that is

either multiplicative,

or an order isomorphism,

or (if \( X \) and \( Y \) are compact) an isometry relative to the sup-norm.

Then \( X \) and \( Y \) are homeomorphic.

The proof of this theorem is spread out over 9.4, 9.5 and 9.6. The point of that is that from the “isomorphism” \( T: C(X) \to C(Y) \) we not only obtain the existence of a homeomorphism \( \tau: Y \to X \), we also describe an explicit relation between \( T \) and \( \tau \); and this relation is different in each of the three situations.

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The following lemma will save us some work.

9.3 Lemma Let \( X \) and \( Y \) be completely regular spaces and let \( \tau: Y \to X \) be such that
\[ f \in C(X) \implies f \circ \tau \in C(Y). \]
Then \( \tau \) is continuous.

Proof Take a net \((y_\alpha)_{\alpha \in A}\) in \( Y \) that converges to a point \( y \). Then for all \( f \in C(X) \) we have \((f \circ \tau)(y_\alpha) \to (f \circ \tau)(y)\), i.e. \( f(\tau(y_\alpha)) \to f(\tau(y))\). Then \( \tau(y_\alpha) \to \tau(y) \) thanks to the complete regularity of \( X \). (See 4.14.) ■

9.4 Theorem (Gelfand, Kolmogorov) Let \( X \) and \( Y \) be realcompact topological spaces. Suppose \( T \) is a linear bijection \( C(X) \to C(Y) \) that is multiplicative, i.e.,
\[ T(fg) = T(f) \cdot T(g) \quad (f, g \in C(X)). \]
Then \( X \) and \( Y \) are homeomorphic. More explicitly: There exists a homeomorphism \( \tau: Y \to X \) for which
\[ T(f) := f \circ \tau \quad (f \in C(X)). \]

Proof (I) First, observe that \( T\mathbb{1}_X = \mathbb{1}_Y \). Indeed, there is an \( f \in C(X) \) with \( Tf = \mathbb{1}_Y \); then \( \mathbb{1}_Y = T(f\mathbb{1}_X) = T(f) \cdot T(\mathbb{1}_X) = \mathbb{1}_Y \cdot T(\mathbb{1}_X) = T(\mathbb{1}_X) \).

(II) Take \( y \in Y \). Then \( \delta^y \circ T \) is a linear function \( C(X) \to \mathbb{R} \) that is multiplicative and maps \( \mathbb{1}_X \) to 1. By the realcompactness of \( X \) there is a \( \tau(y) \in X \) with \( \delta^y \circ T = \delta^{\tau(y)} \). (See 8.2.)

Thus we obtain a map \( \tau: Y \to X \) for which
\[ (\delta^y \circ T)(f) = \delta^{\tau(y)}(f) \quad (f \in C(X), y \in Y), \]
\[ f(\tau(y)) = (Tf)(y) \quad (f \in C(X), y \in Y). \quad (*) \]

Similarly, there is a \( \sigma: X \to Y \) such that
\[ (T^{-1}g)(x) = g(\sigma(x)) \quad (g \in C(Y), x \in X). \quad (**) \]

(III) Take \( x \) in \( X \). For all \( f \in C(X), f(x) = (T^{-1}(Tf))(x) = (Tf)(\sigma(x)) = f \left( \tau(\sigma(x)) \right) \).

As \( C(X) \) separates the points of \( X \), it follows that \( \tau(\sigma(x)) = x \).

In the same way one obtains \( \sigma(\tau(y)) = y \) for all \( y \in Y \). Thus, \( \sigma \) and \( \tau \) are bijections and each other’s inverses. By Lemma 9.3, their continuity follows from (*) and (**). ■

9.5 Theorem (Kaplansky) Let \( X \) and \( Y \) be realcompact topological spaces. Suppose \( T \) is a linear bijection \( C(X) \to C(Y) \) that is an order isomorphism. Then \( X \) and \( Y \) are homeomorphic. Actually, there exist a homeomorphism \( \tau: Y \to X \) and a continuous \( u: Y \to (0, \infty) \) such that
\[ Tf = (f \circ \tau)u \quad (f \in C(X)). \]

(Actually, what Kaplansky proved is not the above but Theorem 9.9.)