Chapter 2

Scaling - Revealing Hierarchy

Details of the self-organization are hierarchical achievements between self-assembly and dissipative structures. The latter is top-down and occurs in a far-from-equilibrium formulation of an open system that involves dissipation of entropy and is characterized by periodic structures, spiral waves, travelling waves, self-similar evolutions, and so forth. The former is, as described in the previous chapter, bottom-up and controlled by the stationary state of the closed system which is involved by condensates, collapses, spikes, quantizations, free energy transmissions, variational structures, and so forth. The control of the total set of stationary states to the global dynamics, however, is not restricted to thermodynamics. This profile is observed widely in mathematical models involved by the mean field hierarchy and sometimes referred to as the nonlinear spectral mechanics. In more precise terms, there is a unified mathematical principle in each mean field hierarchy provided with the underlying physical principle, such as the conservation laws, decrease of the free energy, and so forth.

This chapter describes the quantized blowup mechanism which is one of the leading principle of self-assembly. It arises in self-interacting fluid, turbulence in the context of the propagation of chaos, mean field hierarchy derived from the friction-fluctuation self-interaction in the molecular kinetics, and gauge field concerning condensate of microscopic states. Actually, this profile of quantization is revealed by a blowup analysis which is one of the important products of the method of scaling and is valid even to the higher-space dimension.

2.1. Self-Interacting Continuum

The macroscopic state of particles that constitute the self-interacting fluid is formulated by the system of equations provided with the self-duality between the particle density and the field distribution. Here, the formation of the field is physical and is associated with the Poisson type equation. The stationary state is then described by the nonlinear eigenvalue problem with a non-local term because of the mass conservation, and this problem is provided with the variational structure from the energy conservation. Then the method of scaling detects the critical exponents for mass quantization. This section is devoted to the variational and scaling properties of the systems of self-interacting fluid such as the Euler-Poisson equation and the plasma confinement problem.
2.1.1. Self-Gravitating Fluids

Several fundamental equations of self-interacting fluids are provided with the semi-unfolding-minimality.

Euler-Poisson Equation

With the prescribed velocity \( v = v(x) \in \mathbb{R}^3 \), \( x \in \mathbb{R}^3 \) of the particle, we can define the local flow \( \{ T_t \} \) by (1.2). Thus \( x(t) = T_t x_0 \) indicates the solution to

\[
\frac{dx}{dt} = v(x), \quad x|_{t=0} = x_0.
\]

If \( f(x) \) stands for the observer of particles, then

\[
u(x, t) = f(T_{-t} x)
\]

indicates the distribution of particles detected by it at the time \(-t\). From the semi-group property, it follows that

\[
u(T_t y, t) = f(T_{-T_t} T_t y) = f(y)
\]

for any \( y \in \mathbb{R}^3 \), and, therefore,

\[
0 = \frac{\partial}{\partial t} u(T_t y, t) = \nabla u(T_t y, t) \cdot \frac{\partial}{\partial t} T_t y + u_t(T_t y, t)
= v(T_t y) \cdot \nabla u(T_t y, t) + u_t(T_t y, t),
\]

or

\[
\frac{\partial u}{\partial t} + v \cdot \nabla u = 0 \quad \text{in} \; \mathbb{R}^3 \times (0, T).
\]

Thus we obtain

\[
\frac{Du}{Dt} = 0
\]

for \( u(x, t) = f(T_{-t} x) \).

We call

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla,
\]

the material derivative, which indicates the differentiation in \( t \) of the transported quantity subject to \( v \), and, therefore, the acceleration vector of this fluid is defined by

\[
\frac{Dv}{Dt}.
\]

From this observation, we obtain the Euler equation of motion,

\[
\rho \frac{Dv}{Dt} = \rho F - \nabla p,
\]

where \( \rho \), \( p \), and \( F \) denote the density, the pressure, and the outer force, respectively, and also the equation of continuity

\[
\rho_t + \nabla \cdot (\rho v) = 0
\]