Chapter 4

The Wavelet Transform on Spaces of Type $S$

4.1 Introduction

In this chapter we consider the wavelet transform (3.1.2) for $n = 1$ and write

$$W(\phi) = W\phi(b,a) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ibw} \overline{\psi(aw)} \hat{\phi}(w) dw,$$

(4.1.1)

where $\hat{\phi}$ denotes the Fourier transform of function $\phi$. The wavelet transform on Schwartz space $\mathcal{S}(\mathbb{R})$ was studied in Section 3.2. The spaces of type $S$ play an important role in the theory of linear partial differential equations as intermediate spaces between those of $C^\infty$ and of the analytic functions. The Fourier transform has been studied on the spaces of type $S$ by Friedman [20] and Gel’fand and Shilov [21]. Nevertheless, there exist band-limited wavelets with subexponential decay [15], and also infraexponential decay [76]; see Section 4.6. The aim of the present chapter is to study the wavelet transform (4.1.1) on these spaces.

Let us recall the definitions of these spaces.

**Definition 4.1.1.** The space $S_\alpha(\alpha \geq 0)$ consists of all infinitely differentiable functions $\phi(x)(-\infty < x < \infty)$, satisfying the inequalities

$$\gamma_{k,q}(\phi) := \sup_{x \in \mathbb{R}} |x^k \phi^{(q)}(x)| \leq C_q A^k k^{k\alpha} (k, q = 0, 1, 2, \ldots),$$

(4.1.2)

where the constants $A$ and $C_q$ depend on the function $\phi$. For $k = 0$, the expression $k^{k\alpha}$ is considered to be equal to 1.

**Definition 4.1.2.** The space $S_\beta(\beta \geq 0)$ consists of all infinitely differentiable functions $\phi(x)(-\infty < x < \infty)$, satisfying the inequalities

$$\gamma_{k,q}(\phi) := \sup_{x \in \mathbb{R}} |x^k \phi^{(q)}(x)| \leq C_k B^k q^{q\beta} (k, q = 0, 1, 2, \ldots),$$

(4.1.3)

where the constants $B$ and $C_k$ depend on the function $\phi$. 

\textbf{Definition 4.1.3.} The space $S_\beta^\alpha (\alpha \geq 0, \beta \geq 0)$ consists of all infinitely differentiable functions $\phi(x)$ ($-\infty < x < \infty$), satisfying the inequalities
\begin{equation}
\gamma_{k,q}(\phi) := \sup_{x \in \mathbb{R}} |x^k \phi^{(q)}(x)| \leq CA^kB^qk^{\alpha}q^{\beta}(k,q = 0,1,2,\ldots),
\end{equation}
where the constants $A,B$ and $C$ depend on the function $\phi$.

The spaces of type $S$ are closely interrelated by means of the Fourier transformation; namely, the formulae
\begin{equation}
\tilde{S}_\alpha = S_\alpha, \ \tilde{S}_\beta = S_\beta \text{ and } \tilde{S}_\alpha^\beta = S_\beta^\alpha
\end{equation}
hold.

We shall make use of the following inequalities in our investigation:
\begin{equation}
\frac{q!}{(q-k)!} = k! \left(\frac{q}{k}\right) \leq k! \sum_{k=0}^{q} \left(\frac{q}{k}\right) = k! \ 2^q
\end{equation}
and
\begin{equation}
(m+q)^{(m+q)\beta} \leq m^{\beta} q^{\beta} e^{m\beta} e^{q\beta}
\end{equation}
(See [20, p. 265]). In what follows we shall also need the following similar test function spaces, defined on the upper half-plane $H = \mathbb{R} \times \mathbb{R}_+$, called spaces of type $\tilde{S}$.

\textbf{Definition 4.1.4.} The space $\tilde{S}_\alpha(\mathbb{R} \times \mathbb{R}_+), \tilde{\alpha} = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 \geq 0$ is defined to be the space of all functions $\phi \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that for all $l,s,k,t \in \mathbb{N}_0$,
\begin{equation}
\gamma_{l,s,k,t}(\phi) := \sup_{(b,a) \in \mathbb{R} \times \mathbb{R}_+} |a^l b^s \left(\frac{\partial}{\partial a}\right)^k \left(\frac{\partial}{\partial b}\right)^t \phi(b,a)| \leq C_{l,s,k,t} A_1^l A_2^s B_1^k B_2^t \alpha_1^\alpha_2^\beta,
\end{equation}
where the constants $A_1, A_2$ and $C_{l,s,k,t}$ depend on the testing function $\phi$.

\textbf{Definition 4.1.5.} The space $\tilde{S}_\beta(\mathbb{R} \times \mathbb{R}_+), \tilde{\beta} = (\beta_1, \beta_2), \beta_1, \beta_2 \geq 0$ is defined to be the space of all functions $\phi \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that for all $l,s,k,t \in \mathbb{N}_0$,
\begin{equation}
\gamma_{l,s,k,t}(\phi) := \sup_{(b,a) \in \mathbb{R} \times \mathbb{R}_+} |a^l b^s \left(\frac{\partial}{\partial a}\right)^k \left(\frac{\partial}{\partial b}\right)^t \phi(b,a)| \leq C_{l,s,k,t} B_1^k B_2^t \beta_1^\beta_2,
\end{equation}
where the constants $B_1, B_2$ and $C_{l,s,k,t}$ depend on the function $\phi$.

\textbf{Definition 4.1.6.} The space $\tilde{S}_\alpha(\mathbb{R} \times \mathbb{R}_+), \tilde{\alpha} = (\alpha_1, \alpha_2), \tilde{\beta} = (\beta_1, \beta_2), \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, is defined to be the space of all functions $\phi \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$ such that for all $l,s,k,t \in \mathbb{N}_0$,
\begin{equation}
\gamma_{l,s,k,t}(\phi) := \sup_{(b,a) \in \mathbb{R} \times \mathbb{R}_+} |a^l b^s \left(\frac{\partial}{\partial a}\right)^k \left(\frac{\partial}{\partial b}\right)^t \phi(b,a)| \leq CA_1^l A_2^s B_1^k B_2^t \alpha_1^\alpha_2^\beta \beta_1^\beta_2,
\end{equation}
where the constants $A_1, A_2, B_1, B_2$ and $C$ depend on $\phi$. 