Chapter 5

The Wavelet Transform on Spaces of Type $W$

5.1 Introduction

The spaces of $W$-type were studied by I.M. Gel’fand and G.E. Shilov [22]. They investigated the behaviour of Fourier transformation on the $W$-spaces. Also $W$-spaces are applied to the theory of partial differential equations.

The wavelet transform on Schwartz space $S(\mathbb{R})$ and on spaces of type $S$ has been studied in earlier chapters. In this chapter we recall characterizations of $W$-type spaces introduced in [22] and study the behaviour of continuous wavelet transform (4.1.1) over them.

5.2 The spaces $W_{M,\alpha}, W^{\Omega,\beta}_{M,\alpha}$ and $W^{\Omega,\beta}_{M,\alpha}$

In this section we recall the definitions and properties of Gel’fand- Shilov spaces of $W$-type. We also recall the behaviour of Fourier transformation on the spaces $W_{M,\alpha}, W^{\Omega,\beta}_{M,\alpha}$ and $W^{\Omega,\beta}_{M,\alpha}$ as given in [22].

Let $\mu(\xi) (0 \leq \xi < \infty)$ and $\omega(\eta) (0 \leq \eta < \infty)$ be continuous increasing functions such that $\mu(0) = 0, \mu(\xi) \to \infty$ for $\xi \to \infty$ and $\omega(0) = 0, \omega(\eta) \to \infty$ for $\eta \to \infty$. For $x \geq 0, y \geq 0$, we define

$$M(x) = \int_0^x \mu(\xi) d\xi, \quad M(x) = M(-x), \text{ for } x < 0 \quad (5.2.1)$$

and

$$\Omega(y) = \int_0^y \omega(\eta) d\eta, \quad \Omega(y) = \Omega(-y), \text{ for } y < 0. \quad (5.2.2)$$

The functions $M(x)$ and $\Omega(y)$ are continuous, increasing and convex with $M(0) = 0, M(x) \to \infty$ for $x \to \infty$ and $\Omega(0) = 0, \Omega(y) \to \infty$ for $y \to \infty$. We have the following fundamental convex inequalities,

$$M(x_1) + M(x_2) \leq M(x_1 + x_2), \quad \Omega(y_1) + \Omega(y_2) \leq \Omega(y_1 + y_2). \quad (5.2.3)$$
If the functions $\mu(\xi)$ and $\omega(\eta)$ are mutually inverse, that is, $\mu(\omega(\eta)) = \eta, \omega(\mu(\xi)) = \xi$, then the corresponding functions $M(x)$ and $\Omega(y)$ will be said to be dual in the sense of Young. In this case, we have the following Young inequality

$$xy \leq M(x) + \Omega(y),$$  \hspace{1cm} (5.2.4)

for $x \geq 0, y \geq 0$.

**Definition 5.2.1.** The space $W_{M,\alpha}$, $\alpha > 0$, consists of all complex valued infinitely differentiable functions $\phi(x)$, $(-\infty < x < \infty)$ which for any $\delta > 0$ satisfy

$$\left| \phi^{(q)}(x) \right| \leq C_q e^{-M[\alpha-\delta]x}], \quad q = 0, 1, 2, \ldots$$  \hspace{1cm} (5.2.5)

where positive constants $C_q\delta$ depend on function $\phi(x)$.

**Definition 5.2.2.** The space $W_{\Omega,\beta}$, $\beta > 0$, consists of all entire analytic functions $\phi(z)$, ($z = x + iy \in \mathbb{C}$) which for any $\rho > 0$ satisfy

$$\left| z^k \phi(z) \right| \leq C_k \rho e^{\Omega[(\beta+\rho)y]}, \quad k = 0, 1, 2, \ldots$$  \hspace{1cm} (5.2.6)

where positive constants $C_k\rho$ depend on function $\phi(z)$.

**Definition 5.2.3.** The space $W_{\Omega,\beta}^{\Omega,\alpha}$, $\alpha > 0, \beta > 0$, consists of all entire analytic functions $\phi(z)$, ($z = x + iy \in \mathbb{C}$) which for any $\delta, \rho > 0$ satisfy

$$|\phi(z)| \leq C_{\delta \rho} e^{-M[\alpha-\delta]x] + \Omega[(\beta+\rho)y]},$$  \hspace{1cm} (5.2.7)

where positive constants $C_{\delta \rho}$ depend on function $\phi(z)$.

The following properties are satisfied by the spaces $W_{M,\alpha}, W_{\Omega,\beta}, W_{M,\alpha}^{\Omega,\beta}$ [22,pp.12-24].

1. The operation of differentiation is bounded in $W_{M,\alpha}, W_{\Omega,\beta}, W_{M,\alpha}^{\Omega,\beta}$ and hence is a continuous operation.
2. The operation of multiplication by $x$ in $W_{M,\alpha}$ and multiplication by $z$ in $W_{\Omega,\beta}, W_{M,\alpha}^{\Omega,\beta}$ are bounded and hence are continuous operations.

**Example 1.** Let $M(x) = x^{1/\alpha}(x > 0), \ 0 < \alpha < 1$. Then the space $W_M$ consists of $C^\infty$-functions $\phi$, which satisfy

$$\left| \phi^{(q)}(x) \right| \leq C_q e^{-a|x|^{1/\alpha}},$$

for certain $C_q$ and $a$, depending on $\phi$. This space coincides with the space $S_\alpha$ discussed in chapter 4.