Chapter 3

Theory of Distributions

3.1 Test functions and generalized functions. Regularization of divergent integrals

In the study of boundary integral equations, the theory of distributions naturally comes into use. This theory began with the use of the Dirac delta function by the British physicist P.A.M. Dirac during the 1930s and 1940s. It was found to be extremely useful in solving ordinary and partial differential equations and became very popular, but was rejected by many mathematicians because it was not a classical function and its usage lacked mathematical rigor. In 1950–51, the French mathematician L. Schwartz published Théorie des Distributions [168], making rigorous the theory concerning the usage of the delta function and other distributions. Today, this theory is fundamental in the study of partial differential equations. For a detailed account of the theory of distributions, we refer the reader to [27, 76, 107].

The space $\mathscr{D}(\mathbb{R}^n)$, or briefly $\mathscr{D}$, where

$$\mathscr{D} \equiv C_0^\infty(\mathbb{R}^N),$$

in Chapter 2, is also called the test function space. In $\mathscr{D}$, we define the topology of uniform convergence. We write $\phi_n \to 0$ in $\mathscr{D}$ if

(i) there exists a bounded open set $G$ in $\mathbb{R}^N$ such that supp $\phi_n \subseteq G$ for all $n = 1, 2, \ldots$

(ii) For each multi-index $\alpha$, $0 \leq |\alpha| < \infty$,

$$\lim_{n \to \infty} D^\alpha \phi_n(x) = 0 \text{ uniformly in } \mathbb{R}^N.$$

We write $\phi_n \to \phi$ in $\mathscr{D}$ if $\phi_n - \phi \to 0$ in $\mathscr{D}$.

**Definition 3.1.** A continuous linear functional $T$ on $\mathscr{D}$ is a mapping from $\mathscr{D}$ to $\mathbb{C}$, denoted by $\langle T, \phi \rangle$ for $\phi \in \mathscr{D}$, satisfying
(i) \[ \langle T, c_1 \phi_1 + c_2 \phi_2 \rangle = c_1 \langle T, \phi_1 \rangle + c_2 \langle T, \phi_2 \rangle \quad \forall c_1, c_2 \in \mathbb{C}, \phi_1, \phi_2 \in \mathcal{D}, \]

(ii) \( \phi_n \to 0 \) in \( \mathcal{D} \) implies that \( \langle T, \phi_n \rangle \to 0 \) in \( \mathbb{C} \).

We also call such a continuous linear functional a \textit{distribution} or a \textit{generalized function}. The space of all distributions will be denoted by \( \mathcal{D}'(\mathbb{R}^n) \), or \( \mathcal{D}' \).

The Dirac delta function \( \delta \) is now well defined as a distribution because \( \langle \delta, \phi \rangle \equiv \phi(0) \) is a continuous linear functional on \( \mathcal{D}(\mathbb{R}^N) \).

We may add distributions or multiply them by \( C^\infty \) functions to form new distributions. However, the product of two distributions is not well defined in general.

Every \( L^1_{\text{loc}}(\mathbb{R}^N) \) function \( f \) defines a distribution via

\[ \langle f, \phi \rangle \equiv \int_{\mathbb{R}^N} f(x) \phi(x) \, dx, \quad \forall \phi \in \mathcal{D}. \] 

(3.1)

This suggests the notation

\[ \langle T, \phi \rangle \equiv \int_{\mathbb{R}^N} T(x) \phi(x) \, dx \] 

(3.2)

for a continuous linear functional \( T \) even when \( T \) is not an \( L^1_{\text{loc}} \) function.

Distributions such as \( L^1_{\text{loc}}(\mathbb{R}^N) \) functions can not be assigned values at isolated points. Thus, one can not say that “a distribution \( f \) is equal to zero at \( x_0 \)”. However, one may clearly make a statement that “a distribution \( f \) is equal to zero in a neighborhood \( \mathcal{N} \) of \( x_0 \)”. This will mean that for every \( \phi \in \mathcal{D} \) with support in \( \mathcal{N} \), we have \( \langle f, \phi \rangle = 0 \). Thus, for instance, a distribution \( f \) corresponding to an \( L^1_{\text{loc}}(\mathbb{R}^N) \) function \( f(x) \) vanishes in a neighborhood \( \mathcal{N} \) of \( x_0 \) if \( f(x) \) itself vanishes a.e. in \( \mathcal{N} \). A point \( x_0 \) is called an \textit{essential point} of a distribution \( f \) if there does not exist a neighborhood \( \mathcal{N} \) of \( x_0 \) in which \( f \) vanishes. The closure of the set of all essential points is defined as the \textit{support} of \( f \) and denoted by \( \text{supp} \, f \).

Let us now define the important differentiation operation on distributions. For any \( T \in \mathcal{D} \), we define \( D^\alpha T \) to be a linear functional such that

\[ \langle D^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \phi \rangle \quad \forall \phi \in \mathcal{D}, \]

(3.3)

where \( \alpha \in (\mathbb{Z}^+)^N \) is a given multi-index. It is easy to check that \( D^\alpha T \) itself is now a continuous linear functional, i.e., a distribution.

When \( T \) is a function such that \( D^\beta T \in L^1_{\text{loc}}(\mathbb{R}^N) \) for all \( |\beta| \leq |\alpha| \), then from the notation (3.2) and (3.3), the definition (3.3) amounts to no more than integration by parts. But when \( T \) (is a classical function that) does not admit classical derivatives, (3.3) says that we can still shift all the “burden of differentiability” from \( T \) to \( \phi \) and define a new distribution.