Chapter 1

Introduction to Topological Groups and Semigroups

Notation. We write \( \mathbb{N} \) for the set of positive integers, \( \omega \) for the set of non-negative integers, and \( \mathbb{P} \) for the set of prime numbers. The set of all integers is denoted by \( \mathbb{Z} \), the set of all real numbers is \( \mathbb{R} \), and \( \mathbb{Q} \) stands for the set of all rational numbers.

The symbols \( \tau, \lambda, \kappa \) are used to denote infinite cardinal numbers. A cardinal number \( \tau \) is also interpreted as the smallest ordinal number of cardinality \( \tau \). Each ordinal is the set of all smaller ordinals. Thus, \( \omega \) is both the smallest infinite ordinal number and the smallest infinite cardinal number.

All topologies considered below are assumed to satisfy \( T_1 \)-separation axiom, that is, we declare all one-point sets to be closed. The closed unit interval \([0, 1]\) of the real line \( \mathbb{R} \), with its usual topology, is denoted by \( \mathbb{I} \), and \( \mathbb{S} \) stands for the convergent sequence \( \{1/n : n \in \mathbb{N}\} \cup \{0\} \) with its limit point 0, also taken with the usual topology. We use the symbol \( \mathbb{C} \) to denote the complex plane with the usual sum and product operations, while \( \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \) is the unit circle with center at the origin of \( \mathbb{C} \).

1.1. Some algebraic concepts

In this section we establish the terminology and notation that will be used throughout the book.

In dealing with groups, we will adhere to the multiplicative notation for the binary group operation. In discussions involving a multiplicative group \( G \), the symbol \( e \) will be reserved for the identity element of \( G \).

We are very much concerned with groups in this course. For many purposes, however, it is natural and convenient considering semigroups. A semigroup is a non-void set \( S \) together with a mapping \((x, y) \rightarrow xy\) of \( S \times S \) to \( S \) such that \( x(yz) = (xy)z \) for all \( x, y, z \) in \( S \). That is, a semigroup is a non-void set with an associative multiplication. Given an element \( x \) of a semigroup \( S \), one inductively defines

\[
x^2 = xx, \quad x^3 = xx^2, \quad \ldots, \quad x^{n+1} = xx^n,
\]

for every \( n \in \mathbb{N} \). The associativity of multiplication in \( S \) implies the equality \( x^n x^m = x^{n+m} \) for all \( x \in S \) and \( n, m \in \mathbb{N} \).

An element \( e \) of a semigroup \( S \) is called an identity for \( S \) if \( ex = x = xe \) for every \( x \in S \). Not every semigroup has an identity (see items 4) and 6) of Example 1.1.1). However, if a semigroup \( S \) has an identity, then it is easy to see that this identity is unique. Whenever
we use the symbol \( e \) without explanation, it always stands for the identity of the semigroup under consideration.

A semigroup with identity is called monoid. An element \( a \) of a monoid \( M \) is said to be invertible if there exists an element \( b \) of \( M \) such that \( ab = e = ba \). Note that if \( a \) is an invertible element of a monoid \( M \), then the element \( b \in M \) such that \( ab = e = ba \) is unique. Indeed, suppose that \( ab = e, ba = e, ac = e, \) and \( ca = e \). Then we have

\[
   c = ec = (ba)c = b(ac) = be = b.
\]

This fact enables us to use notation \( a^{-1} \) for such an element \( b \) of \( M \). We also say that \( b \) is the inverse of \( a \). It is clear that \((a^{-1})^{-1} = a\) for each invertible element \( a \in M \). Further, one can define negative powers of an invertible element \( a \in M \) by the rule \( a^{-n} = (a^{-1})^n \), for each \( n \in \mathbb{N} \). It is a common convention to put \( a^0 = e \). We leave to the reader a simple verification of the equality \( a^n a^m = a^{n+m} \) which holds for all \( n, m \in \mathbb{Z} \).

If every element \( a \) of a monoid \( M \) is invertible, then \( M \) is called a group.

Let \( S \) be a semigroup. For a fixed element \( a \in S \), the mappings \( x \mapsto ax \) and \( x \mapsto xa \) of \( S \) to itself are called the left and right actions of \( a \) on \( S \), and are denoted by \( \lambda_a \) and \( \varrho_a \), respectively.

If \( G \) is a group, the mapping \( x \mapsto x^{-1} \) of \( G \) onto itself is called inversion. Left and right actions of every element \( a \in G \) on \( G \) are, in this case, bijections. They are called left and right translations of \( G \) by \( a \).

**Example 1.1.1.** Each of the following is a semigroup but not a group.

1) The set \( \mathbb{Z} \) of all integers with the usual multiplication.
2) The set \( \mathbb{Q} \) of all rational numbers with the usual multiplication.
3) The set \( \mathbb{R} \) of all real numbers with the usual multiplication.
4) The set of all positive real numbers with the usual addition in the role of the product operation.
5) The set \( \mathbb{N} \), in which the product of \( x \) and \( y \) is defined as \( \max\{x, y\} \).
6) The set \( \mathbb{N} \), in which the product of \( x \) and \( y \) is defined as \( \min\{x, y\} \).
7) Any set \( S \) with \( |S| > 1 \), where the product \( xy \) is defined as \( y \).
8) Any set \( S \) with \( |S| > 1 \), where the product \( xy \) is defined as \( x \).
9) The set \( S(X, X) \) of all mappings of a set \( X \) to itself with the composition of mappings in the role of multiplication, where \( |X| > 1 \).

In items 4) and 6) of the above example, the corresponding semigroups have no identity. The semigroups in 1)–3), 5), and 9) are monoids.

Now we present a few standard examples of groups.

**Example 1.1.2.** Each of the following is a group:

1) The set \( \mathbb{Z} \) of all integers with the usual addition in the role of multiplication.
2) The set \( \mathbb{Q} \setminus \{0\} \) of all non-zero rational numbers with the usual multiplication.
3) The set \( \mathbb{R} \setminus \{0\} \) of all non-zero real numbers with the usual multiplication.
4) The set of all positive real numbers with the usual multiplication.
5) The set \( \{0, 1\} \) with the binary operation defined as follows:

\[
0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1, \quad 1 + 1 = 0.
\]

This group is denoted by \( \mathbb{Z}(2) \) or by \( D \); it is called the cyclic group of two elements, or the two-element group. More generally, for an integer \( n > 1 \), let \( \mathbb{Z}(n) = \{0, 1, \ldots, n - 1\} \)