Chapter 10

Actions of Topological Groups on Topological Spaces

In this short chapter we introduce an important topic of continuous actions of topological groups on topological spaces. No attempt is made at a systematic treatment of the subject; this would require a separate book. Some such books already exist (see, in particular, [530, 86]). Our goal is much more modest — to give the reader just the flavour of the topic by establishing several important results on dyadicity or similar properties of compact spaces in this context. One of these results concerns compact $G_δ$-sets in quotient spaces of $ω$-balanced topological groups. Even the corollary dealing with the case of the quotient space itself is extremely interesting and highly non-trivial. Another theorem provides a deep insight into the topological structure of compact spaces admitting a continuous transitive action of an $ω$-narrow topological group. In fact, all compact spaces just mentioned have the following strong property — they are Dugundji spaces. Our arguments require several topological facts which usually do not form a part of standard courses on general topology, so the first section of the chapter familiarizes the reader with the concepts of Dugundji spaces, 0-soft mappings, and nearly open mappings. We also develop further the techniques involving inverse spectra (in Chapter 4 we have already made the first steps in this direction). We also introduce some basic concepts and elementary results on actions of topological groups on topological spaces.

10.1. Dugundji spaces and 0-soft mappings

A compact space $X$ is called Dugundji if for every zero-dimensional compact space $Z$ and every continuous mapping $f : A \to X$, where $A$ is a closed subset of $Z$, there exists a continuous mapping $g : Z \to X$ extending $f$.

\[
\begin{array}{c}
A \\
\downarrow^f
\end{array}
\quad
\begin{array}{c}
Z \\
\downarrow^g
\end{array}
\quad
\begin{array}{c}
X
\end{array}
\]

In the proposition below we establish two basic properties of Dugundji spaces.

**Proposition 10.1.1.** The class of Dugundji spaces has the following properties:

a) every compact metrizable space is Dugundji;

b) the product of an arbitrary family of Dugundji spaces is Dugundji.
Proof. a) Let \( f : A \to X \) be a continuous mapping, where \( X \) is a compact metrizable space and \( A \) is a closed subset of a zero-dimensional compact space \( Z \). Consider a mapping \( F : Z \to \text{Exp}(X) \) of \( Z \) to the family \( \text{Exp}(X) \) of closed subsets of \( X \) defined by \( F(z) = \{ f(z) \} \) for each \( z \in A \) and \( F(z) = X \) for \( z \in Z \setminus A \). Then the mapping \( F \) is lower semicontinuous, so Theorem 4.1.1 implies that there exists a continuous selection \( g : Z \to X \) for \( F \). It follows from the definition of \( F \) that \( g(z) = f(z) \), for each \( z \in A \), so \( g \) is a continuous extension of \( f \) over \( Z \). Hence, \( X \) is Dugundji.

b) Suppose that \( X = \prod_{i \in I} X_i \) is the product of a family \( \{ X_i : i \in I \} \) of Dugundji spaces. Since each space \( X_i \) is compact, the product space \( X \) is also compact. Let \( f : A \to X \) be a continuous mapping, where \( A \) is a closed subset of a zero-dimensional compact space \( Z \). For every \( i \in I \), consider the mapping \( f_i = p_i \circ f \), where \( p_i : X \to X_i \) is the natural projection. Since \( f_i : A \to X_i \) is a continuous mapping to the Dugundji space \( X_i \), it admits a continuous extension \( g_i : Z \to X_i \). Let \( g \) be the diagonal product of the family \( \{ g_i : i \in I \} \). Then the mapping \( g : Z \to X \) is continuous and \( g|A = f \); thus, \( X \) is Dugundji.

Combining items a) and b) of the above proposition, we deduce the following:

Corollary 10.1.2. The product of an arbitrary family of second-countable compact spaces is Dugundji.

Let us now establish a simple but very important fact:

Theorem 10.1.3. [R. Haydon] Every Dugundji space is dyadic.

Proof. Let \( X \) be an arbitrary Dugundji space of weight \( \tau \geq \omega \). By Theorem 4.1.5, we can find a closed subset \( A \) of the space \( D^\tau \), where \( D = \{0, 1\} \) is the discrete two-point space, and a continuous onto mapping \( f : A \to X \). Since \( X \) is Dugundji and \( D^\tau \) is compact and zero-dimensional, \( f \) can be extended to a continuous mapping \( g : D^\tau \to X \). Clearly, \( g(D^\tau) = g(A) = f(A) = X \), so \( X \) is dyadic.

A continuous mapping \( f : X \to Y \) is called 0-soft if for every zero-dimensional compact space \( Z \), every continuous mapping \( g : Z \to Y \) and a continuous mapping \( h : A \to X \) of a closed subset \( A \) of \( Z \) satisfying \( g|A = f \circ h \), there exists a continuous mapping \( \varphi : Z \to X \) extending \( h \) which makes the following diagram commutative.

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & Z \\
\downarrow{h} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
\]

It follows immediately from the above definition that a compact space \( X \) is Dugundji if and only if a mapping of \( X \) to a singleton (i.e., a constant mapping) is 0-soft. A priori, 0-soft mappings are not assumed to be onto. However, every 0-soft mapping of compact spaces is surjective.

Proposition 10.1.4. If \( f : X \to Y \) is a 0-soft mapping of compact spaces \( X \) and \( Y \), then \( f(X) = Y \).

Proof. By Theorem 4.1.5, there exists a zero-dimensional compact space \( Z \) and a continuous onto mapping \( g : Z \to Y \). Let \( y \in f(X) \) be an arbitrary point. Choose \( x \in X \) and \( z \in Z \) such that \( f(x) = y = g(z) \), and put \( A = \{ z \} \). Define a mapping \( h : A \to X \)