Chapter 4

Some Special Classes of Topological Groups

Chapter 4 is one of the main in the book. It develops, in various directions, the central theme that we have already touched upon in previous chapters — how the presence of a synchronized algebraic structure influences properties of a topology. We consider below, from this point of view, a series of most important topological properties such as compactness (Sections 4.1 and 4.2), Čech-completeness and featheriness (Section 4.3), the $P$-property, extremal disconnectedness (Sections 4.4 and 4.5), the Fréchet–Urysohn property and weak first countability (Section 4.7). The choice of the above mentioned properties is justified not only by their remarkable role in General Topology, but by the fact that each of them has already been a subject of deep and unexpected results in topological algebra.

In Section 4.1 we establish, using only techniques of General Topology, that every compact topological group is a dyadic compactum, and that the cellularity of every compact group is countable. In Section 4.2 we show, by a direct elementary construction, that every non-metrizable compact group contains a topological copy of the generalized Cantor discontinuum $D^m$. These statements have many corollaries for the topological structure of compact groups.

As it is well known, completeness type properties play the fundamental role in General Topology and its applications. In Section 4.3 we consider one of these properties, Čech-completeness, in its connections with topological groups. It turns out that, unlike the case of general topological spaces, Čech-completeness of topological groups implies paracompactness and Raïkov completeness of the groups. We also show that Čech-complete groups and feathered groups are naturally related by open perfect homomorphisms to metrizable groups.

Section 4.4 is devoted to $P$-groups, that is, to topological groups in which every $G_δ$-subset is open. Lindelöf $P$-spaces, in many respects, behave as compact Hausdorff spaces. We establish that every Lindelöf $P$-group is Raïkov complete, and that every continuous homomorphism of a Lindelöf $P$-group onto another Lindelöf $P$-group is open.

In Section 4.5, devoted to extremally disconnected topological groups, we give an elementary proof of a well-known theorem on algebraic structure of extremally disconnected topological groups saying that every such group contains an open subgroup consisting of elements of order $\leq 2$. It is established that every extremally disconnected topological skew field is discrete. We also discuss the almost 40 years old problem: Is there in $ZFC$ alone a non-discrete extremally disconnected topological group? At the end of the section we present an important example of a non-discrete maximal (hence, extremally disconnected)
topological group whose construction requires a weak form of Martin’s Axiom abbreviated to $p = c$.

In Section 4.6 we have a look at the role of perfect mappings in the theory of topological groups from another angle. This allows to obtain original results on connections between some properties of subspaces of topological groups such as Čech-completeness, featheriness, paracompactness, metrizability, and similar properties of the group itself.

Section 4.7 treats certain delicate convergence properties in topological groups: Fréchet–Urysohn property, weak first countability, and bisequentiality. It turns out that in topological groups these properties are transformed greatly, and connections between them are considerably strengthened. We prove, in particular, that every Fréchet–Urysohn topological group is strongly Fréchet–Urysohn, that the product of a Fréchet–Urysohn topological group with a first-countable space is a Fréchet–Urysohn space, and that every weakly first-countable topological group, as well as every bisequential topological group, is metrizable.

In the chapter we formulate a series of open problems. The mastership of the material of this chapter can open good perspectives for research in various main stream directions of topological algebra.

4.1. Ivanovskij–Kuz’minov Theorem

In this section we prove the celebrated theorem of Ivanovskij[259] and Kuz’minov[287] that every compact topological group is a dyadic compactum. Recall that a dyadic compactum is a compact Hausdorff space that can be represented as an image of the generalized Cantor discontinuum $D^\tau$ under a continuous mapping, where $D = \{0, 1\}$ is the two-point discrete space and $\tau$ is a cardinal. It is well known that every metrizable compact space is a continuous image of the Cantor set $D^\omega$[165, 3.2.B]. Thus, all metrizable compacta are dyadic. However, not all compact spaces are dyadic, as we will see below. The proof of Ivanovskij–Kuz’minov’s theorem is based on an important theorem of E. A. Michael on selections. We start with a proof of the later theorem, and then present a certain technique involving well-ordered inverse spectra of compact spaces, playing a crucial role in the proof of the main theorem.

If $M$ is a space, then $\text{Exp}(M)$ stands for the set of all closed non-empty subsets of $M$, and $\mathcal{F}(M)$ is the set of all non-empty subsets of $M$. A mapping $q$ of a topological space $X$ into the set $\mathcal{F}(M)$ is called lower semicontinuous if, for each open subset $V$ of $M$, the set $V_q$ of all $x \in X$ such that $q(x) \cap V \neq \emptyset$ is open in $X$. Particularly important is the case when $q(x) \in \text{Exp}(M)$ for each $x \in X$, that is, when each $q(x)$ is a non-empty closed subset of $M$. Then we, of course, say that $q$ is a lower semicontinuous mapping of $X$ into $\text{Exp}(M)$.

Let $X$ and $M$ be some spaces and $q$ a mapping of $X$ into $\mathcal{F}(M)$. A mapping $f$ of $X$ to $M$ is called a selection for $q$ if $f(x) \in q(x)$, for each $x \in X$. If, in addition, the mapping $f$ of $X$ into $M$ is continuous, $f$ is said to be a continuous selection for the mapping $q$.

We call a mapping $f$ locally constant if, for every point $x \in X$, there exists an open neighbourhood $W$ of $x$ such that $f(y) = f(x)$, for each $y \in W$. Of course, a locally constant function is continuous.

Now we are ready to formulate a version (not the strongest one) of Michael’s Selection Theorem:

Ivanovskij–Kuz’minov Theorem