Chapter 5

Cardinal Invariants
of Topological Groups

The definition of the notions of topology and topological space, based on the axiomatic approach, is of necessity of a purely set-theoretic nature. Indeed, a topology is just a family of sets satisfying certain axioms. Not so many elementary and natural properties of sets can be formulated without recourse to special, more complicated, structures or tools of mathematical logic. The most important, and almost the only such property is the cardinality of a set. So no wonder that in General Topology cardinal invariants, that is, characteristics of spaces preserved by homeomorphisms and formulated in terms of cardinal numbers and of families of sets, play a central, almost universal, role. Cardinal invariants measure the size of the space in various ways, the local behaviour of the space, and, most importantly, they are used to bring to light specific features of the space. When we consider continuous mappings, it is important to know which cardinal invariants do not increase under certain natural restrictions on these mappings. When studying products of spaces, it is most useful, whatever our main interest may be, to know how certain cardinal invariants behave under the Tychonoff product operation. Some of the questions of this kind are quite deep and difficult, and the work on them has generated much of the progress not only in General Topology, but in Set Theory and in Mathematical Logic as well. To make the point, it is enough to mention the following question. Suppose that the Souslin number of a space $X$ is countable. Is the Souslin number of the square $X \times X$ of $X$ countable? This question, which is so easy to formulate and understand, is intimately related to the famous Souslin Conjecture and to Martin’s Axiom, coined in Mathematical Logic, both of which have so many consequences in Topology and Analysis.

Cardinal invariants of a somewhat different kind play a fundamental role in algebra also; one can refer to the cardinality of a basis of a vector space, to the $p$-rank and torsion-free rank of an Abelian group (see Section 9.9), and to similar concepts. So we should expect that cardinal invariants must have a prominent role in topological algebra. Examples of that were seen in preceding chapters; it suffices to mention the Birkhoff–Kakutani theorem on the metrizability of first-countable topological groups, the theorem on the metrizability of compact groups of countable tightness, or the theorem stating that the cellularity of every compact topological group is countable.

This chapter is devoted to a deeper, and more systematic, study of cardinal invariants of topological groups, with detours into the realm of paratopological groups. We introduce a variety of cardinal invariants, some of them of mixed, topological and algebraic nature, and we study relationships between them. One of the main points of interest is to clarify how the presence of a “synchronous” algebraic structure influences the behaviour of purely
topological cardinal invariants in the new ambient, how the relationship between them changes.

One of the ways to understand the essence and the role of a topological cardinal invariant is to consider the class of all objects (topological spaces, topological groups, and so on) satisfying certain restriction on the value of this invariant, and to study the categorical properties of the class so obtained, that is, to investigate whether the class is closed under the product operation, whether it is preserved by various classes of mappings, whether it is hereditary.

We study such questions below and, although the theory is obviously far from complete, we present, along with simple basic facts, certain deep and delicate results of a very general nature. We hope that some of these results will serve as corner stones for the emerging theory.

5.1. More on embeddings in products of topological groups

We already know when a topological group $G$ is topologically isomorphic to a subgroup of a product of second-countable groups — according to Gurău’s theorem, this happens if and only if $G$ is $\omega$-narrow (see Theorem 3.4.23). It is very natural to introduce more general classes of groups by taking subgroups of arbitrary products of some “nice” topological groups. Given a class $\mathcal{P}$ of topological groups, it is also natural to try to find an internal characterization of the subgroups of the groups in $\mathcal{P}$. This is still an open problem for the class $\mathcal{P}$ of Lindelöf topological groups. On the other hand, subgroups of compact topological groups are precisely the precompact groups, by Corollary 3.7.17. The following definition generalizes the concepts of precompact and $\omega$-narrow groups.

Let $\tau$ be an infinite cardinal. A left topological group $G$ is called $\tau$-narrow if, for every neighbourhood $U$ of the identity in $G$, there exists a subset $K \subset G$ with $|K| \leq \tau$ such that $KU = G$.

We collect several simple properties of $\tau$-narrow topological groups in the following proposition.

**Proposition 5.1.1.**

a) Every subgroup of a $\tau$-narrow topological group is $\tau$-narrow.
b) If $\pi: G \to H$ is a continuous homomorphism of a $\tau$-narrow left topological group $G$ onto a left topological group $H$, then $H$ is $\tau$-narrow.
c) The topological product of arbitrarily many $\tau$-narrow left topological groups is $\tau$-narrow.
d) If $G$ is a dense $\tau$-narrow subgroup of a topological group $H$, then $H$ is also $\tau$-narrow.

Let us mention that in the case $\tau = \omega$, item a) of Proposition 5.1.1 coincides with Theorem 3.4.4, item b) coincides with Proposition 3.4.2, item c) is exactly Proposition 3.4.3, and item d) is Theorem 3.4.9. Since the general case of $\tau$-narrow groups does not substantially differ from that of $\omega$-narrow groups, we leave Proposition 5.1.1 without proof.

The above proposition implies that $\mathbb{R}^\omega$ is an $\omega$-narrow group that fails to be $\sigma$-compact. In fact, many non-closed subgroups of the groups $\mathbb{R}$ and $\mathbb{T}$ are not $\sigma$-compact; all of them are $\omega$-narrow, by a) of Proposition 5.1.1. The next example shows even more.