Expansions in Fadle–Papkovich Functions in a Strip.
Theory Foundations

M. D. Kovalenko* and T. D. Shulyakovskaya**
Schmidt Institute of Physics of the Earth, Russian Academy of Sciences,
B. Gruzinskaya 10, Moscow, 123995 Russia
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Abstract—We consider a boundary-value problem of elasticity for a half-strip with free longitudinal
sides and some conditions at the end. We present a general scheme for solving the problem in the
form of explicit expansions in Fadle–Papkovich functions and study the basis properties of systems
of Fadle–Papkovich functions. The theory is based on the Borel transform in the class of quasi-entire
functions of exponential type.

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1. INTRODUCTION

The present paper deals with expansions in Fadle–Papkovich functions (in the domestic literature,
they are also called homogeneous solutions) in the boundary-value problem of elasticity for the half-strip
{\Pi: x \geq 0, |y| \leq 1} with free sides y = \pm 1. One of the last surveys of this field contains over 750 selected
references for a period of more than 100 years [1].

In the first part of the present paper, we consider the expansions of a single function over a system of
Fadle–Papkovich functions. We construct biorthogonal function systems and present explicit formulas
for the expansion coefficients. The obtained expansions are called Lagrange series expansions (or, for
short, Lagrange expansions). They are analogues of Fourier series expansions and play the same role in
solving boundary-value problems as the Fourier series in the Fadlon–Ribier solutions.

The systems of Fadle–Papkovich functions are not orthogonal and do not have function systems
that are orthogonal to them on an interval (the strip end). That is why it is impossible to obtain explicit
expressions for the coefficients of the desired expansions by using classical concepts of a function basis.
But it is possible to construct biorthogonality relations for the Fadle–Papkovich functions by using the
following generalization of the classical notion of biorthogonality. The construction of function systems
biorthogonal to systems of exponentials defined on the interval |y| \leq 1 is based on the notion of duality
existing between the class of entire functions of exponential type and the class of functions that are
analytic in the plane of the complex variable x + iy cut along the segment |y| \leq 1 of the imaginary axis.
This duality is established by the Borel transform in the class of entire functions of exponential type [2, 3].

The Fadle–Papkovich functions defined on the interval |y| \leq 1 can be considered as a generalization
of the system of exponentials, which leads to a new type of duality, namely, between the class of quasi-
entire functions of exponential type 1 and the class of functions that are analytic and unique in the plane
of the complex variable x + iy cut along the segment |y| \leq 1 of the imaginary axis and along an arbitrary
ray issuing from the origin. Such a duality is established by the Borel transform in the class of quasi-
entire functions of exponential type. This transform was introduced in 1935 in [4]. The properties of this
transform were studied in [5] to an extent that is necessary in applications of the theory of elasticity.
Some of the results obtained there are used in the present paper.

In the second part of this paper, we present a scheme for solving the boundary-value problem itself in
the form of explicit expansions in Fadle–Papkovich functions for one of the possible versions of boundary
conditions at the strip end. We study the relationship between the results obtained in the paper and the known expansions based on the Papkovich orthogonality relations.

2. LAGRANGE EXPANSIONS

2.1. Biorthogonal Function System

It is known that the solution of the boundary-value problem of elasticity in the strip II with free longitudinal sides can be represented as series in Fadle–Papkovich functions as follows:

\[ \sigma_x(x, y) = (1 + \mu) \sum_{k=1}^{\infty} a_k \lambda_k \sigma_x(\lambda_k, y)e^{-\lambda_k x} + \bar{a}_k \lambda_k \sigma_x(\lambda_k, y)e^{-\bar{\lambda}_k x} + C_1, \]

\[ \sigma_y(x, y) = (1 + \mu) \sum_{k=1}^{\infty} a_k \lambda_k \sigma_y(\lambda_k, y)e^{-\lambda_k x} + \bar{a}_k \lambda_k \sigma_y(\lambda_k, y)e^{-\bar{\lambda}_k x}, \]

\[ \tau_{xy}(x, y) = (1 + \mu) \sum_{k=1}^{\infty} a_k \lambda_k^2 \tau_{xy}(\lambda_k, y)e^{-\lambda_k x} + \bar{a}_k \lambda_k^2 \tau_{xy}(\lambda_k, y)e^{-\bar{\lambda}_k x}, \]

\[ U(x, y) = \sum_{k=1}^{\infty} a_k U(\lambda_k, y)e^{-\lambda_k x} + \bar{a}_k U(\lambda_k, y)e^{-\bar{\lambda}_k x} + \frac{C_0}{2(1 + \mu)} + \frac{C_1 x}{2(1 + \mu)}, \]

\[ V(x, y) = \sum_{k=1}^{\infty} a_k V(\lambda_k, y)e^{-\lambda_k x} + \bar{a}_k V(\lambda_k, y)e^{-\bar{\lambda}_k x} - \frac{\mu C_1 y}{2(1 + \mu)} \quad (\text{Re} \lambda_k > 0), \]

where \( U(x, y) = G u(x, y) \), \( V(x, y) = G v(x, y) \), \( u(x, y) \) and \( v(x, y) \) are the longitudinal and transverse displacements, \( G \) is the shear modulus, and \( C_0 \) and \( C_1 \) are arbitrary constants (corresponding to an elementary solution).

In the case symmetric deformation of the strip, the Fadle–Papkovich functions have the form

\[ \sigma_x(\lambda_k, y) = (\sin \lambda_k - \lambda_k \cos \lambda_k) \cos(\lambda_k y) - \lambda_k y \sin \lambda_k \sin(\lambda_k y), \]

\[ \sigma_y(\lambda_k, y) = (\sin \lambda_k + \lambda_k \cos \lambda_k) \cos(\lambda_k y) + \lambda_k y \sin \lambda_k \sin(\lambda_k y), \]

\[ \tau_{xy}(\lambda_k, y) = \cos \lambda_k \sin(\lambda_k y) - y \sin \lambda_k \cos(\lambda_k y), \]

\[ U(\lambda_k, y) = \left( \frac{1 - \mu}{2} \sin \lambda_k - \frac{1 + \mu}{2} \lambda_k \cos \lambda_k \right) \cos(\lambda_k y) - \frac{1 + \mu}{2} \lambda_k y \sin \lambda_k \sin(\lambda_k y), \]

\[ V(\lambda_k, y) = \left( \frac{1 + \mu}{2} \lambda_k \cos \lambda_k + \sin \lambda_k \right) \sin(\lambda_k y) - \frac{1 + \mu}{2} \lambda_k y \sin \lambda_k \cos(\lambda_k y), \]

\[ \tau_{xy}(\lambda_k, \pm 1) = \sigma_y(\lambda_k, \pm 1) = 0 \quad (k \geq 1). \]

The numbers \( \lambda_k \) form the set \( \{ \pm \lambda_k, \pm \bar{\lambda}_k \}_{k=1}^{\infty} = \Lambda \) of all complex zeros of the following entire function of exponential type 2:

\[ L(\lambda) = \lambda + \sin \lambda \cos \lambda. \]

We use expressions (2.1) to satisfy, for example, the normal stress \( \sigma(y) \) and tangential stress \( \tau(y) \) given on the half-strip end and obtain the problem of determining the coefficients \( a_k \ (k \geq 1) \) of the expansions

\[ \sigma(y) = (1 + \mu) \sum_{k=1}^{\infty} a_k \lambda_k \sigma_x(\lambda_k, y) + \bar{a}_k \lambda_k \sigma_x(\lambda_k, y), \]

\[ \tau(y) = (1 + \mu) \sum_{k=1}^{\infty} a_k \lambda_k^2 \tau_{xy}(\lambda_k, y) + \bar{a}_k \lambda_k^2 \tau_{xy}(\lambda_k, y). \]

The function \( \sigma(y) \) is even and the function \( \tau(y) \) is odd. Moreover, \( \sigma(y) \) must be self-balanced (i.e., the integral of this function from \(-1\) to \(1\) is zero), because all functions \( \sigma_x(\lambda_k, y) \) are also self-balanced.