On the Precession of the Elliptic Mode Shape of a Circular Ring Owing to Nonlinear Effects

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Abstract—The Foucault pendulum, which maintains the plane of its vibrations in inertial space, loses this property as soon as the trajectory ceases to be flat. If the pendulum end circumscribes an elliptic trajectory instead of a straight line segment, then this ellipse precesses in the same direction as the material point circumscribes the ellipse itself. In this case, the angular velocity of the ellipse precession is proportional to its area and can be explained by the nonlinearity of the equations of vibrations of a mathematical pendulum [1].

A similar phenomenon takes place in an elastic inextensible ring, which is a representative of the “generalized Foucault pendulum” family [1]. If a standing wave is excited in an immovable ring, then this wave is immovable with respect to the ring only in the case of zero quadrature, but if the quadrature is nonzero, then the standing wave precesses with respect to the ring with a velocity proportional to the quadrature value.

As in the case of the classical pendulum, this phenomenon can be explained by the nonlinearity of the ring regarded as an oscillatory system.

In the present paper, we obtain an explicit formula for calculating the angular velocity of such a precession.

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Following [2, 3], we derive the equations of vibrations of a thin inextensible ring without the assumption that the vibration amplitude is small. In the plane xy, consider a ring in an arbitrary deformed state (Fig. 1).

We use the following variables and notation. The independent variable s determines the position of the considered point on the ring and is the length of the arc between the initial point on the ring and the point under study. The polar coordinates of this point, which determine its position in the plane xy, are denoted by r and θ. The angle Φ is an auxiliary angle, which is further required to calculate the energy potential of the deformed state of the ring.

It is convenient to use the equations of motion in Lagrangian form, for which one needs to write out the potential energy density, the kinetic energy density, and the constraint equation. First, we derive the constraint equation. To this end, we calculate the length of the arc of the curve representing the ring midline between the points s and s + Δs, where Δs is a small increment. From the triangle formed by the increments Δϕ, Δr, and Δs in Fig. 2, we obtain

\[(Δs)^2 = (Δr)^2 + (rΔθ)^2 = \left(\frac{∂r}{∂s}\right)^2 + \left(\frac{r∂θ}{∂s}\right)^2(Δs)^2,\]

which implies the constraint equation in the form

\[(r')^2 + (rθ')^2 = 1.\]  (1)

We use the prime to denote differentiation with respect to the independent variable determining the position of a point on the ring.

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The kinetic energy density has the form
\[ T = \frac{1}{2} \rho S (r'^2 + r^2 \dot{\theta}^2), \] (2)
where \( S \) is the ring cross-section area. The angle \( \Phi \) (Fig. 1) determines the position of the normal to the ring axial line at the point under study. The derivative of this angle along the deformed ring, which is denoted by \( \Phi' \), characterizes the curvature of the deformed ring at this point. The bending stresses appear in the ring together with variations in its curvature given by the formula \( \Phi' - \Phi'_0 \), where \( \Phi'_0 \) denotes the derivative of \( \Phi \) at the considered point for the undeformed ring. The potential energy density can therefore be written as
\[ \Pi = \frac{1}{2} EI (\Phi' - \Phi'_0)^2. \] (3)
Thus, the density of the Lagrangian function becomes
\[ L(\dot{\theta}, r, \dot{r}, \Phi') = \frac{1}{2} S \rho [(r^2 + r^2 \dot{\theta}^2) - EI (\Phi' - \Phi'_0)^2]. \] (4)

The variable \( \Phi \) is not a generalized coordinate and should be expressed in terms of \( r \) and \( \theta \). The angle between the position vector direction at a given point of the curve and the normal to the curve at this