BRIEF COMMUNICATIONS

Depth of Functions of k-Valued Logic in Finite Bases

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Abstract—Realization of functions of the k-valued logic by circuits is considered over an arbitrary finite complete basis B. Asymptotic behavior of the Shannon function \( D_B(n) \) of the circuit depth over B is examined. The value \( D_B(n) \) is the minimal depth sufficient to realize every function of the k-valued logic of n variables by a circuit over B. It is shown that for each natural \( k \geq 2 \) and for any finite complete basis B there exists a positive constant \( \alpha_B \) such that \( D_B(n) \sim \alpha_B n \) for \( n \to \infty \).

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In this paper we consider the depth of functions of the k-valued logic (\( k \geq 2 \)) in realization by circuits of functional elements over an arbitrary basis. A basis means an arbitrary finite set of functions of the k-valued logic such that any function of the k-valued logic can be realized by superpositions of functions from this set. The circuit depth is the maximal number of elements in oriented chains connecting some input of the circuit with its output. The depth of a function \( f \) of the k-valued logic over a basis B means the minimal depth of circuits realizing the function \( f \) over the basis B. We denote this value by \( D_B(f) \). For each basis B we introduce the Shannon function \( D_B(n) \) characterizing the maximal depth of functions of n variables and determined by the equality \( D_B(n) = \max f D_B(f) \), where the maximum is taken over all functions \( f \) of the k-valued logic dependent on \( n \) variables.

It was established [1] that in the case of the two-valued logic (\( k = 2 \)) the relation \( D_B(n) \sim \beta n, n \to \infty \), holds for any basis B, here \( \beta = (\log_2 m)^{-1} \) and \( m \) is the maximal number of essential variables of functions from the basis B.

In this paper we prove the following

**Theorem 1.** For an arbitrary finite complete basis B of functions of the k-valued logic (\( k \geq 2 \)) there exists a positive constant \( \alpha_B \) such that the relation \( D_B(n) \sim \alpha_B n \) holds for \( n \to \infty \).

Let B be an arbitrary basis of functions of the k-valued logic. For any integer nonnegative number \( d \), by \( N_B^*(d) \) we denote the greatest number of essential variables of functions admitting a realization over the basis B by a circuit of depth not exceeding \( d \). It is not difficult to prove

**Lemma 1.** The inequality \( N_B^*(m+1) \leq N_B^*(m)N_B^*(l) \) is valid for any integer nonnegative numbers \( m \) and \( l \).

Formulate the following well-known fact from the mathematical analysis (see, e.g., [2]).

**Lemma 2.** Let \( \{a_n\} \) be a sequence of nonnegative numbers such that for any natural numbers \( m \) and \( l \) the inequality \( a_{m+l} \leq a_m + a_l \) holds. Then the limit \( \lim_{n \to \infty} \frac{a_n}{n} \) exists.

**Lemma 3.** The sequence \( \{a_d\} \), where \( a_d = \frac{\log_2 N_B^*(d)}{d}, d = 1, 2, 3, \ldots \), has the limit for \( d \to \infty \).

It is not difficult to prove that the inequality \( \lim_{d \to \infty} \frac{\log_2 N_B^*(d)}{d} > 0 \) is valid. Assume

\[ \alpha_B = \left( \lim_{d \to \infty} \frac{\log_2 N_B^*(d)}{d} \right)^{-1}. \]

Using cardinality arguments (see, e.g., [3]) one can obtain the following assertion.

**Lemma 4.** The inequality \( D_B(n) > \alpha_B n (1 - \varepsilon) \) is valid for any \( \varepsilon, \varepsilon > 0 \), and sufficiently large \( n \), in the case the portion of functions \( f(x_1, \ldots, x_n) \) such that \( D_B(f) < \alpha_B n (1 - \varepsilon) \) tends to zero with the growth of \( n \).

Denote the set \( \{0, \ldots, k - 1\} \) by \( E_k \). An arbitrary function \( h(x_1, \ldots, x_n) : \{0,1\}^n \to E_k \) is called a Boolean quasifunction. Note that a Boolean function is a particular case of a Boolean quasifunction. We say that the function \( g(x_1, \ldots, x_n) \) of the k-valued logic generates the Boolean quasifunction \( f(x_1, \ldots, x_n) \) if the equality \( f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n) \) holds for any collection \( (a_1, \ldots, a_n) \) of zeros and ones. For any Boolean quasifunction \( f \), by \( D_B^f \) we denote \( \min g D_B(g) \), where the minimum is taken over all functions \( g \) of the k-valued logic generating the Boolean quasifunction \( f \). For any natural \( n \), the notation \( D_B^f(n) \) means \( \max f D_B^f(f) \), where the maximum is taken over all Boolean quasifunctions \( f \) dependent on \( n \) variables.

**Lemma 5.** The relation \( D_B^f(n) \leq (\log_2 2)\alpha_B n + o(n) \) holds for \( n \to \infty \).
Lemma 6. There exists a positive constant $c$ such that for any natural $n$ the inequality $D_B(n) \leq cn$ is valid.

Associate each Boolean quasifunction $f$ with some function $f^*$ of the $k$-valued logic (taken arbitrarily) generating the Boolean quasifunction $f$ and satisfying the condition $D_B(f^*) = D_B(f)$.

Lemma 7. The relation $D_B(n) \leq \alpha_B n + o(n)$ holds for $n \to \infty$.

Proof. Consider an arbitrary collection $\tilde{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ from $E_k^n$. Represent the collection $\tilde{\gamma}$ in the form $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_s)$, where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are collections of length $R = \lceil \log_2 n \rceil$ (the last collection can have a lesser length, denote its length by $R'$). It is easy to see that $s = \lceil n/\log_2 n \rceil$.

Let $a$ be an arbitrary natural number and $b = \lceil \log_2 k^R \rceil$. We encode arbitrary sets from $E_k^n$ by collections of zeros and ones of length $b$. It is easy to see that for each $a$ there exists a certain one-to-one encoding $K_{a,b}$ (such that different codewords correspond to different sets from $E_k^n$). For an arbitrary set $\delta$ from $E_k^n$, by $(\beta_1^a, \ldots, \beta_b^a)$ we denote the corresponding codeword under the encoding $K_{a,b}$.

Thus, the collections $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{s-1}, \tilde{\gamma}_s$ are associated with the codewords

$$((\beta_1^1, \ldots, \beta_{\log_2 k^R}^1), \ldots, (\beta_1^{s-1}, \ldots, \beta_{\log_2 k^R}^{s-1}), (\beta_1^s, \ldots, \beta_{\log_2 k^R}^s)).$$

Associate each set $\tilde{\gamma}$ from $E_k^n$ with the collection

$$\tilde{\beta}_{\tilde{\gamma}} = ((\beta_1^1, \ldots, \beta_{\log_2 k^R}^1), \ldots, (\beta_1^{s-1}, \ldots, \beta_{\log_2 k^R}^{s-1}, \beta_1^s, \ldots, \beta_{\log_2 k^R}^s))$$

of zeros and ones. The length of such collection does not exceed $s \lceil \log_2 k^R \rceil$. Estimate this value from above. We have

$$s \lceil \log_2 k^R \rceil = \lceil n/\log_2 n \rceil \lceil \log_2 k^R \rceil \leq \lceil n/\log_2 n \rceil \lceil \log_2 k \log_2 n \rceil < \lceil n/\log_2 n \rceil \lceil \log_2 k \log_2 n \rceil + 1$$

$$< (n/\log_2 n + 1)(\log_2 n + 1) = (n/\log_2 n + 1)(\log_2 k \log_2 n + (\log_2 k + 1))$$

$$= (\log_2 k)n + (\log_2 k + 1)(n/(\log_2 n) + \log_2 k \log_2 n + (\log_2 k + 1)) = (\log_2 k)n + o(n).$$

Note that different sets $\tilde{\gamma}, \tilde{\sigma}$ from $E_k^n$ are associated with different collections $\tilde{\beta}_{\tilde{\gamma}}, \tilde{\beta}_{\tilde{\sigma}}$.

Let $h = h(x_1, x_2, \ldots, x_n)$ be an arbitrary function of the $k$-valued logic. Consider the functions of the $k$-valued logic

$$g_1^1(y_1, \ldots, y_R), g_2^1(y_1, \ldots, y_R), \ldots, g_1^{\lceil \log_2 k^R \rceil}(y_1, \ldots, y_R),$$

$$g_1^{s-1}(y_1, \ldots, y_R), g_2^{s-1}(y_1, \ldots, y_R), \ldots, g_1^{\lceil \log_2 k^R \rceil}(y_1, \ldots, y_R),$$

such that for any $i$ and $j$ and for any set $\tilde{\gamma}$ from $E_k^n$ we have the relation $g_i^j(\tilde{\gamma}_i) = \beta_i^{j_i}$. Due to Lemma 6, the depth of each such function does not exceed $\log(\log_2 n)$.

The expression

$$t_h = t_h(z_1^1, \ldots, z_R^1, \ldots, z_{s-1}^1, z_R^{s-1}, z_1^s, \ldots, z_R^s)$$

defines some Boolean quasifunction (taken arbitrarily) satisfying the following condition: for any set $\tilde{\gamma}$ from $E_k^n$, the following equality holds:

$$t_h(\beta_1^1, \ldots, \beta_{\log_2 k^R}^1), \ldots, \beta_1^{s-1}, \ldots, \beta_{\log_2 k^R}^{s-1}, \beta_1^s, \ldots, \beta_{\log_2 k^R}^s) = h(\tilde{\gamma}).$$

By virtue of Lemma 5 and relation (1), we get

$$D_B(t_h^n) = D_B(t_h) \leq D_B(s(\lceil \log_2 k^R \rceil)) \leq \alpha_B n + o(n).$$

The function $h = h(x_1, x_2, \ldots, x_n)$ can be represented in the form

$$h(x_1, x_2, \ldots, x_n) = t_h^1(g_1^1(x_1, \ldots, x_R), \ldots, g_1^{\lceil \log_2 k^R \rceil}(x_1, \ldots, x_R), \ldots, g_1^{s-1}(x_{(s-2)} R+1, \ldots, x_{(s-1)} R), \ldots, \ldots, g_1^s(x_{(s-1)} R+1, \ldots, x_{n}).$$

Therefore, the following relations are valid for an arbitrary function $h$ of $n$ variables:

$$D_B(h) \leq D_B(t_h^n) + \max_{i,j} D_B(g_i^j) \leq \alpha_B n + o(n) + C_1 \lceil \log_2 n \rceil = \alpha_B n + o(n).$$

Lemma 7 is proved.