The Explicit Form of the Bertrand Metric

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Abstract—The problem of explicit form of the metric of revolution on Bertrand’s Riemannian manifolds in particular coordinates is solved. Connections with earlier results of M. Santoprete are discussed.

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1. INTRODUCTION

In [1], the concept of Bertrand Riemannian manifolds was introduced. Such manifolds arise as configuration spaces in the problem on a point motion along closed trajectories in a potential field. In the general case, a Bertrand manifold is a two-dimensional Riemannian manifold $S \approx I \times S^1$ (here $I \subset \mathbb{R}$ is an interval) endowed with the following revolution metric:

\[
\begin{pmatrix}
1 & 0 \\
0 & f^2(r)
\end{pmatrix}
\]

in the coordinate $(r, \varphi \mod 2\pi)$ and such that there exists at least one central potential (referred to as the Bertrand potential) that provides the closeness of some class of trajectories for a point motion in a given potential field. Those closed orbits that are non-circular are given by the periodic functions $r = r(\varphi)$ with the minimal positive period $\Phi = \frac{2\pi}{\beta}$, where $\beta \in \mathbb{Q}_{>0}$ is a rational constant referred to as a Bertrand constant.

In this paper Bertrand Riemannian manifolds mean a slightly wider class of manifolds obtained by changing the closeness property for a class of non-circular orbits by the requirement that the orbits from that class are given by the periodic functions $r = r(\varphi)$ with the minimal positive period $\Phi = \frac{2\pi}{\beta}$, where $\beta \in \mathbb{R}_{>0}$ is some constant not necessarily rational.

It was shown in [1] that Bertrand Riemannian manifolds without equators (i.e., such that $f'(r) \neq 0$ on $I$) are exactly the manifolds whose Riemannian metric can be written in some explicit form (see (3) below) in some coordinates $(\theta, \varphi \mod 2\pi)$ such that $\theta = \theta(r)$, and either $\mu = \frac{1}{2}$, or $\mu = \frac{1}{2}$.

On the other hand, Santoprete [2] proved that the function $f$ standing in the metric of a Bertrand manifold without equators has to satisfy some differential equation (see (2) below), where $\beta$ is the Bertrand constant. Analyzing this important differential equation, Santoprete proved that such Riemannian manifolds admit at most two central Bertrand potentials up to additive and multiplicative constants.

We say that a revolution Riemannian manifold with metric (3) belongs to the first type if $ff'' + f'^2 \equiv const$ (this is equivalent to the constancy of the Riemannian curvature of the manifold), otherwise we say that the manifold belongs to the second type.

This paper presents the proof of the equivalence of these two conditions mentioned above (the possibility to represent the metric in some special from in some coordinates, and the differential condition of Santoprete). Namely, the following theorem is valid.

**Theorem.** Let $S \approx (a, b) \times S^1$ be a two-dimensional manifold with the coordinates $(r, \varphi \mod 2\pi)$ and Riemannian metric $ds^2 = dr^2 + f^2(r)d\varphi^2$, where $f(r)$ is a smooth function on $(a, b)$ and $f'(r) \neq 0$ on $(a, b)$. Then if the function $f(r)$ satisfies the equation

\[
\beta^4 - 5(-ff'' + f'^2)\beta^2 - 5ff'' + f'^2 + 4f''f^2 - 3f''f'f^2 + 4f'^2 = 0
\]

for some constant $\beta \in \mathbb{R}_{>0}$, then this Riemannian manifold $(S, ds^2)$ is a Bertrand manifold in the sense of [1], i.e., there exist coordinates $(\theta, \varphi \mod 2\pi)$ such that $\theta = \theta(r)$ and the metric in this coordinates has the form

\[
ds^2 = \frac{d\theta^2}{(\theta^2 + c - t\theta^{-2})^2} + \frac{d\varphi^2}{\mu^2(\theta^2 + c - t\theta^{-2})^2},
\]

for some constant $c$. 

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where $\mu, c, \text{ and } t$ are some real constants, $\mu > 0$. In this case $\mu \in \left\{ \frac{1}{2}, \frac{2}{3} \right\}$. Moreover, if $ff'' - f''^2 \equiv \text{const} \ (\text{revolution Riemannian manifold of the first type})$, then $ff'' - f''^2 \equiv -\frac{\beta^2}{\mu^2}$ for $i \in \{1, 2\}$, $t = 0, \mu = \frac{1}{3}$; and if $ff'' - f''^2 \neq \text{const} \ (\text{manifold of the second type})$, then $t \neq 0, \mu = \frac{2}{3}$.

Conversely, for any Bertrand manifold without equators in the sense of [1] (i.e., a Riemannian manifold with metric (3)) the function $f(r)$ obtained from expression (3), i.e., such that $f^2(r(\theta)) = \mu\left(\theta^2 + c - \mu\theta^{-2}\right)$, where the dependence $r = r(\theta)$ on $\theta$ is determined from the equality $r'(\theta) = (\theta^2 - \theta^{-2})^{-1}$, satisfies Equation (2) with the constant $\beta := \frac{2}{\mu}$ for $t \neq 0$, and for any constant $\beta \in \left\{ \frac{1}{\mu}, \frac{2}{\mu} \right\}$ for $t = 0$.

In general, it is interesting to study two-dimensional revolution Riemannian manifolds and, in particular, Bertrand Riemannian manifolds from the viewpoint the theory of integrable system. A Hamiltonian system on Bertrand manifolds is completely integrable in the Liouville sense. Since explicit forms for the metrics of these Riemannian manifolds and for the Bertrand potentials are known, it turns out to be possible to calculate their bifurcation diagrams and labeled molecules, i.e., complete invariants of Liouville equivalence for these systems. (The details concerning Hamiltonian systems and labeled molecules (Fomenko–Zieschang invariants) can be found, e.g., in [3–5].)

2. EXPLICIT FORM FOR RIEMANNIAN METRIC ON BERTRAND MANIFOLDS

Prove the theorem. To do that, first we prove the lemma on an explicit form of the function $f$ in the coordinate $\theta = \theta(r)$.

**Lemma 1.** Under the conditions of the direct part of the theorem, let the function $f = f(r)$ satisfies equation (2). Then there exist a unique constant $\mu > 0$ and a function $\theta = \theta(r)$ such that $\theta'(r) = \frac{1}{\mu^2} \frac{1}{f^2(r)}$ and the equality

$$f^2(r(\theta)) = \frac{1}{\mu^2(\theta^2 + c - \mu\theta^{-2})}$$

holds, where $c, t$ are real constants. In this case $\mu \in \left\{ \frac{1}{2}, \frac{2}{3} \right\}$. Moreover, if $ff'' - f''^2 \equiv \text{const} \ (\text{manifold of the first type})$, then $ff'' - f''^2 \equiv -\frac{\beta^2}{\mu^2}$ for $i \in \{1, 2\}$, $t = 0, \mu = \frac{1}{3}$; if $ff'' - f''^2 \neq \text{const} \ (\text{manifold of the second type})$, then $t \neq 0, \mu = \frac{2}{3}$.

In order to prove Lemma 1, we first prove the following lemma on the equivalence of equation (2) for the function $f(r)$ obtained by Santoprete in [2] and the equation for the function $f(r(\theta))$ used in [1] (see equation (5) below) for a Riemannian manifold of the second type. The equivalence of equations means here the following: if the function $f(r)$ is a solution to the first equation, then the function $f(r(\theta))$, where $\theta'(r) = \beta^2 r^{-\frac{1}{\beta - 1}}$, is a solution to the second equation, and vice versa, if the function $g(\theta) := f(r(\theta))$, where $\theta(r)$ was defined above, is a solution to the second equation, then the function $f(r)$ is a solution to the first equation.

**Lemma 2.** Let $\beta > 0$ and a function $f = f(r) > 0$ such that $ff'' - f''^2 \neq -\frac{\beta^2}{4}$, $ff'' - f''^2 \neq -\beta^2$, and $ff'' - f''^2 \neq 0$ be given, and the inequalities $\theta'_i$ are supposed to be valid for all $r \in (a, b)$. Let the change of variables $\theta = \theta(r)$ satisfy the relation $\theta'(r) = \beta^2 r^{-\frac{1}{\beta - 1}}$ and let $r = r(\theta)$ be the inverse change of variables, where $\mu = \frac{1}{\beta} > 0, i = 1, 2$, is an arbitrary constant. Then the third order differential equation

$$\beta^3 - 5(-f''f + f'^2)\beta^2 - 5f''f f'^2 + 4f'^2f^2 - 3f''^2f^2 f'^2 + 4f'^2 = 0 \quad (4)$$

for the function $f(r)$ is equivalent to the existence of constants $c_1, c_2 \in \mathbb{R}, c_1 \neq 0$ such that the function $f(r(\theta))$ satisfies the following first order differential equation:

$$\frac{\beta^2}{t^4} \frac{d^2 f(r(\theta))}{d\theta^2} = c_1(\theta + c_2)^{-3} - \frac{1}{4}(\theta + c_2). \quad (5)$$

**Proof.** We prove Lemma 2 step by step where each step is an equivalent transformation of the equations.

**Step 1.** We use the following notation: $h(r) := f(r)f''(r) - f'^2(r)$. Using it, rewrite equation (4) in the form

$$\beta^4 + 5h(r)\beta^2 - 3f(r)f'(r)h'(r) + 4h^2(r) = 0. \quad (6)$$

**Step 2.** Now, introduce the function $\eta(\theta) := \frac{1}{(\mu\beta)^2} \frac{f'(r(\theta))}{f^2(r(\theta))} = \frac{\beta^2}{t^4} \frac{f(r(\theta))}{f^2(r)}$.

We can verify that the following equations are valid:

$$h(r) = \beta^2 \eta_0(\theta(r)), \quad h'(r) = \frac{\beta^4}{t^4} \frac{\eta_0(\theta(r))}{f^2(r)}.$$