General Categorical Framework for Topologically Free Normed Modules
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Abstract—It is shown that the strict projectivity of normed modules is a special case of the projectivity in a rigged category. A criterion is given for a bornological space to be a base for a free object in the corresponding category. A certain class of categories is indicated where each projective object is a retract of a free object.

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1. INTRODUCTION

The concept of a projective module is principal in topological homologies. There are different kinds of projectivity, but all of them are obtained from the general categorial concept of projectivity under an appropriate choice of a rigged category, see [1]. The so-called strict or topological projectivity is one of the most known types of projectivity. In this paper we show that this kind of projectivity of normed modules appears if the rigging is the forgetful functor mapping into the category of so-called bornological spaces. The class of bornological spaces being bases for free objects in the corresponding categories is described.

The so-called freedom-loving categories are most convenient for studying rigged categories, in this case each object of an auxiliary category is a base for a free object of the main category. All projective objects in those categories are retracts of free objects. It is interesting that the category of normed spaces with rigging in bornological spaces is not freedom-loving, but each projective object in it is  a retract of a free object. A general categorial interpretation of this fact is given in the end of the paper.

For a normed algebra $A$, by $A^+$ we denote its initialization and $e_+$ denotes the unit of the algebra $A^+$. We consider the category $\mathbf{A-mod}$ of normed modules (its morphisms are bounded module morphisms), the category $\mathbf{A-mod}$ of Banach modules (its morphisms are the same as in $\mathbf{A-mod}$), and the category Nor of normed spaces (its morphisms are bounded operators). By $\otimes_p$ we denote the (non-augmented) projective tensor product of normed spaces or normed modules.

2. BORNOLOGICAL SPACES

In some cases it is convenient to consider this category as rigging. We describe the corresponding concepts and constructions.

Definition 1 (cf. [2]). Let a set $E$ be given. A family $B$ of its subsets is called a bornology on $E$ if $B$ satisfies the following properties:
1) if $A \in B$, $B \in B$, then $A \cup B \in B$;
2) if $A \subseteq B$ and $B \in B$, then $A \in B$;
3) for any point $x \in E$ we have $\{x\} \in B$.

A set $E$ together with a bornology $B$ fixed on it is called a bornological space. Subsets of $E$ belonging to the bornology $B$ are referred to as bounded sets.

Definition 2. A base of a bornology $B$ on $E$ is a subsystem $B_0$ in $B$ such that each element of $B$ is contained in some element of $B_0$.

Definition 3. Let $E$ and $F$ be two bornological spaces and $u: E \to F$ be a mapping. We say that $u$ is bounded if it takes any bounded subset of $E$ to a bounded subset of $F$.

Thus, we consider some category denoted by $\text{Born}$. Its objects are bornological spaces and its morphisms are bounded mappings.

Let $E$ be a normed space. Then its subsets bounded with respect to the norm (i.e., subsets contained in some balls) form a bornology on $E$, and each continuous operator $u: E \to F$ acting between normed spaces is bounded with respect to this bornology. Thus, one can define the forgetful functor $\square: \text{Nor} \to \text{Born}$. 
Define 4. The bornology on a set $E$ consisting of all finite subsets of $E$ is called the discrete bornology. The bornology consisting of all subsets of $E$ is called the antidiscrete topology.

The terms "discrete" and "antidiscrete" are borrowed from topology. To justify them, we formulate the following evident statement.

**Proposition 1.** Let $E$ be a bornological space. Then
1) all mappings from $E$ to an arbitrary bornological space $F$ are bounded if and only if $E$ is discrete;
2) all mappings from an arbitrary bornological space $F$ to $E$ are bounded if and only if $E$ is antidiscrete.

A similar result is valid for topological spaces and continuous mappings.

3. PROJECTIVITY IN RIGGED CATEGORY

Let $K$, $L$ be some categories. Recall that a functor $\square: K \to L$ is said to be faithful (or exact) if for any two distinct morphisms $\varphi_1$, $\varphi_2$ in $K$ the relation $F(\varphi_1) \neq F(\varphi_2)$ is valid.

**Definition 5** (cf. [1]). A rigging of a category $K$ is said to be the faithful functor $\square: K \to L$. A rigged category is a category $K$ endowed with the rigging $\square: K \to L$. Strictly speaking, it is the pair $(K, \square)$.

Let $(K, \square)$ be a rigged category, $\square: K \to L$, $F$ be an object of $K$, and $M$ be an object from $L$.

**Definition 6** (cf. [1, 3]). An object $F$ is called a free object with the base $M$ if it has a universal arrow $j$, i.e., there exists a morphism $j: M \to \square F$ such that for any object $X \in K$ and any morphism $\varphi: M \to \square X$ there exists a unique morphism $\psi: F \to X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\square F & \xrightarrow{\psi} & \square X \\
\downarrow & & \downarrow \\
M & \xrightarrow{j} & \varphi
\end{array}
$$

**Definition 7** (see [1]). A morphism $\varphi$ in $K$ is said to be admissible epimorphism if $\square \varphi$ is a retraction.

Let we be given with an object $P \in K$, a morphism $\sigma: Y \to X$ in $K$, and a morphism $\varphi: P \to X$. Recall that the problem of finding a morphism $\psi: P \to Y$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Y & \xrightarrow{\psi} & P \\
\downarrow & \searrow \sigma & \downarrow \varphi \\
X & \to & X
\end{array}
$$

is called the lifting problem $(P, \sigma, \varphi)$.

The lifting problem is said to be admissible if $\sigma$ is an admissible epimorphism.

**Definition 8** (cf. [4]). An object $P \in K$ is called projective if any admissible lifting problem $(P, \sigma, \varphi)$ is solvable.

Let $\square_N: A-\text{mod} \to \text{Nor}$ denote the forgetful functor from the modules category to the category of normed spaces.

We formulate the following evident statement.

**Proposition 2.** The family of traditionally projective modules in $A-\text{mod}$ (i.e., the modules being projective in the sense of [1]) coincides with the set of projective modules in the rigged category $(A-\text{mod}, \square_N)$.

**Definition 9.** A module $P$ is said to be strictly (or topologically) projective if any lifting problem $(P, \sigma, \varphi)$ with an open morphism $\sigma$ is solvable.

This definition can be transferred word-by-word to the case of Banach modules.

Let objects of a category $K$ be linear spaces (generally speaking, endowed with additional structure) and let morphisms be (some) linear operators. Then the forgetful functor $\square_L: K \to \text{Lin}$ gives a rigging for $K$.

**Proposition 3.** The set of strictly projective Banach modules is equal to the set of projective objects in $(A-\text{mod}, \square_L)$.

**Proof.** We have to verify that the admissible morphisms in $(A-\text{mod}, \square_L)$ are exactly the open morphisms.

In accordance with Banach theorem, any surjective operator on Banach spaces is open. Therefore, if a morphism $\varphi$ is a retraction in $\text{Lin}$, then it is open. Conversely, an open morphism is surjective and hence it is a retraction in $\text{Lin}$. $\square$