Exact Lower Estimate of the Upper Limit of the Ratio of the Sum of Sine Series with Monotone Coefficients to its Majorant

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Abstract—An unimprovable lower estimate of the upper limit of the ratio of the sum of sine series with monotone coefficients to its majorant is obtained.

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We study the asymptotic behavior of the sum of the sine series \( g(b, x) = \sum_{k=1}^{\infty} b_k \sin kx \) for \( x \to 0^+ \); the sequence of coefficients \( b = \{b_k\}_{k \in \mathbb{N}} \) monotonically tends to zero, i.e.,

\[
b_1 > 0, \ b_{k+1} \leq b_k \ \forall k \in \mathbb{N}, \ \lim_{k \to \infty} b_k = 0. \tag{1}
\]

Denote the class of all sequences (1) by \( \mathfrak{M} \). It was proved in [1] that the function \( v(b, x) = x \sum_{k=1}^{m(x)} kb_k \), \( m(x) = \lfloor \pi / x \rfloor \), is a majorant of \( g(b, x) \) on the interval \( (0, \pi) \) for any sequence \( b \in \mathfrak{M} \). Therefore, for all \( b \in \mathfrak{M} \) the following inequality is valid:

\[
\overline{t}(b) = \lim_{x \to 0^+} \frac{g(b, x)}{v(b, x)} \leq 1. \tag{2}
\]

The asymptotic estimate \( g(b, x) - v(b, x) = O\left(x^3 \sum_{k=1}^{m(x)} k^3 b_k\right), \ x \to 0^+ \), derived by Telyakovskii [2] and the Hartman–Wintner theorem [3] stating that

\[
\lim_{x \to 0^+} \frac{g(b, x)}{x} = \begin{cases} 
\sum_{k=1}^{\infty} kb_k, & \text{for } \sum_{k=1}^{\infty} kb_k < +\infty; \\
+\infty, & \text{for } \sum_{k=1}^{\infty} kb_k = +\infty
\end{cases}
\]

allow us to prove the limit relation

\[
b_k = O\left(k^{-2}\right), \ b = \{b_k\} \in \mathfrak{M} \Rightarrow \lim_{x \to 0^+} \frac{g(b, x)}{v(b, x)} = 1. \tag{3}
\]

Relations (2), (3) imply \( \max \{\overline{t}(b) \mid b \in \mathfrak{M}\} = 1 \).

What is the infimum of \( \overline{t}(b) \) taken over all sequences \( b \in \mathfrak{M} \)? Aljančić, Bojačić, and Tomić [4] proved that if a positive, decreasing, tending to zero, and slowly varying at infinity [5, p. 10] convex function \( b \) is given on the ray \([1, +\infty)\), then

\[
\sum_{k=1}^{\infty} b(x) \sin kx \sim x^{-1} b^{-1}(x^{-1}), \ x \to 0.
\]

Since in this case \( x \sum_{k=1}^{m(x)} kb(x) \sim (1/2) x b^{-1}(x^{-1}) m^2(x) \sim (\pi^2 / 2) x^{-1} b^{-1}(x^{-1}), \ x \to 0 \), then

\[
\lim_{x \to 0^+} \frac{g\left(\{b(n)\}, x\right)}{v\left(\{b(n)\}, x\right)} = \frac{2}{\pi^2}.
\]

Therefore, the inequality \( \inf \{\overline{t}(b) \mid b \in \mathfrak{M}\} \leq 2\pi^{-2} \) holds. We proved that the equality actually holds here.
Theorem 1. The following equality holds: \( \min \{ \mathcal{T}(b) \mid b \in \mathfrak{M} \} = 2\pi^{-2} \).

Due to the theorem of Aljančić, Bojač, and Tomic cited above, it is sufficient to verify that \( \mathcal{T}(b) \geq 2\pi^{-2} \) \((\forall b \in \mathfrak{M})\) to prove the theorem. We obtain a more fine result.

Theorem 2. For any sequence \( b \in \mathfrak{M} \) there exists a sequence of positive numbers \( \{x_p\} \), \( \lim_{p \to \infty} x_p = 0 \), such that for all \( p \in \mathbb{N} \) the inequalities \( g(b, x_p) > 2\pi^{-2} v(b, x_p) \) are valid.

In its turn, Theorem 2 follows from Theorem 3.

Theorem 3. Given arbitrary \( b \in \mathfrak{M} \) and \( n \in \mathbb{N} \), there exists a natural number \( N > n \) such that the following inequality holds:

\[
\int_{\pi/2N}^{\pi/2n} (g(b, x) - 2\pi^{-2} v(b, x)) \, dx > 0.
\]

The existence of the sequence \( \{x_p\}_{p \in \mathbb{N}} \) stated in Theorem 2 is deduced from Theorem 3 by the method of mathematical induction. In fact, take \( n = 1 \) and, according to Theorem 3, choose natural \( N > 1 \) so that (4) is valid. Assume \( n_1 = N \). If the sequence of numbers \( \{n_k\}_{k=1}^p \) has been already constructed, then for \( n = n_p \) we again choose \( N \) so that condition (4) holds, denote this number by \( n_{p+1} \). As the result, we obtain the sequence of numbers \( n_p \) such that

\[
\int_{\pi/2n_{p+1}}^{\pi/2n_p} (g(b, x) - 2\pi^{-2} v(b, x)) \, dx > 0.
\]

On each interval \((\pi/(2n_{p+1}), \pi/(2n_p))\) we take the point \( x_p \) where \( g(b, x_p) - 2\pi^{-2} v(b, x_p) > 0 \). Obviously, the sequence \( \{x_p\} \) constructed here satisfies all the conditions of Theorem 2.

We premise the proof of Theorem 3 with four lemmas.

Lemma 1. Let \( n, \nu \in \mathbb{N} \), \( \{\beta_k\}_{k \in \mathbb{N}} \in \mathfrak{M} \). In this case

\[
S = \sum_{k=(n(4\nu-3)+1)}^{n(4\nu+1)} \beta_k \cos \left( \frac{\pi k}{2n} \right) \leq 0.
\]

Proof. Denote \( A = n(4\nu - 3) \) and divide the sum \( S \) into two following parts:

\[
S = \sum_{k=A+1}^{A+2n} \beta_k \cos \left( \frac{\pi k}{2n} \right) + \sum_{k=A+2n+1}^{A+4n} \beta_k \cos \left( \frac{\pi k}{2n} \right) = 2n \sum_{j=1}^{2n} \beta_{A+j} \cos \left( \frac{\pi (A+j)}{2n} \right) + 2n \sum_{j=1}^{2n} \beta_{A+2n+j} \cos \left( \frac{\pi (A+2n+j)}{2n} \right) = 2n \left( \beta_{A+j} - \beta_{A+j} \right) \sin \left( \frac{\pi j}{2n} \right).
\]

Since \( \sin(\pi j/(2n)) > 0 \) for \( 1 \leq j < 2n, \beta_{A+j} \leq \beta_{A+j} \), then all summands in the latter sum are not positive and hence \( S \leq 0 \). The lemma is proved.

Lemma 2. Let \( n, \nu \in \mathbb{N} \), \( \{\beta_k\}_{k \in \mathbb{N}} \in \mathfrak{M} \). In this case the following inequalities are valid:

\[
\sum_{k=2N}^{\infty} \beta_k \cos \left( \frac{\pi k}{2n} \right) \geq -N \beta_{2N}, \quad \left| \sum_{k=2N}^{\infty} \beta_k \cos \left( \frac{\pi k}{2n} \right) \right| \leq 2n \beta_{2N}.
\]

Proof. Divide the first sum into two following parts:

\[
\sum_{k=2N}^{3N} \beta_k \cos \left( \frac{\pi k}{2n} \right) + \sum_{k=3N+1}^{\infty} \beta_k \cos \left( \frac{\pi k}{2n} \right) = \sum_{k=2N}^{3N-1} \beta_k \cos \left( \frac{\pi k}{2n} \right) - \sum_{q=N+1}^{\infty} \beta_{2N+q} \cos \left( \frac{\pi q}{2n} \right).
\]

(5)

Taking into account the double inequality \( -1 \leq \cos(\pi k/(2N)) < 0, 2N \leq k < 3N \), and also the non-negativity and monotonicity of \( \{\beta_k\} \), we have

\[
\sum_{k=2N}^{3N-1} \beta_k \cos \left( \frac{\pi k}{2n} \right) \geq - \sum_{k=2N}^{3N-1} \beta_k \geq -N \beta_{2N}.
\]

(6)