The Contact Problem for an Elastic Half-Plane in the Case of a Transonic Load Velocity

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Received March 14, 2012; in final form, April 26, 2013

Abstract—The stress-strain state of an elastic half-space under the action of a moving load is studied. The problem is solved analytically. The solution obtained is analyzed for the case of a transonic load velocity.

DOI: 10.3103/S0027133013040018

The transonic velocities are typical for the interaction of space debris particles with the protective shields of spacecrafts or with their surfaces. The problems of such contact interactions are studied in [1] in a static formulation and in [2] in a dynamic formulation. In addition to the discussion of a number of dynamic contact problems, Galin’s book [2] contains a detailed review of works on this subject. Mathematical methods for solving contact interaction problems are considered in [3, 4]. In this paper we assume that an elastic half-space is loaded by a rigid stamp and that the motion is steady in the coordinate system associated with this stamp.

Figure 1 shows an observer-fixed coordinate system (x′ o′ y′). In this system the plane-parallel motion of an elastic medium satisfies the following wave equations:

\[
\begin{align*}
\frac{\partial^2 \varphi}{\partial t^2} &= a^2 \left( \frac{\partial^2 \varphi}{\partial x'^2} + \frac{\partial^2 \varphi}{\partial y'^2} \right), \\
\frac{\partial^2 \psi}{\partial t^2} &= b^2 \left( \frac{\partial^2 \psi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial y'^2} \right), \\
a &= \sqrt{(\lambda + 2\mu)/\rho}, \quad b = \sqrt{\mu/\rho}.
\end{align*}
\]

(1)

Here \( \varphi(x', y', t') \) is the potential of longitudinal waves, \( \psi(x', y', t') \) is the potential of transverse waves, \( a \) is the velocity of the longitudinal waves, \( b \) is the velocity of the transverse waves, \( \rho \) is the density of the medium, \( \lambda \) and \( \mu \) are Lamé’s elastic constants, and \( t' \) is time. The displacement field is expressed in terms of the potentials by the following Lamé formulas:

\[
\begin{align*}
ux &= \frac{\partial \varphi}{\partial x'}, \quad uy = \frac{\partial \psi}{\partial y'}, \\
ux &= \frac{\partial \varphi}{\partial x'} - \frac{\partial \psi}{\partial y'}.
\end{align*}
\]

(2)

Substituting (2) into (1) and into Hooke’s law, we come to the following expressions for stresses:

\[
\begin{align*}
\sigma_{xx} &= \lambda \left( \frac{\partial^2 \varphi}{\partial x'^2} + \frac{\partial^2 \varphi}{\partial y'^2} \right) + \mu \left( \frac{\partial^2 \varphi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial y'^2} \right), \\
\sigma_{yy} &= \lambda \left( \frac{\partial^2 \varphi}{\partial x'^2} + \frac{\partial^2 \varphi}{\partial y'^2} \right) + \mu \left( \frac{\partial^2 \varphi}{\partial y'^2} - \frac{\partial^2 \psi}{\partial x'^2} \right), \\
\sigma_{xy} &= \mu \left( \frac{2}{\partial x'} \frac{\partial^2 \varphi}{\partial y'^2} + \frac{\partial^2 \psi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial y'^2} \right).
\end{align*}
\]

(3)

In the coordinate system (xoy) associated with the moving load (Fig. 1), the motion is assumed to be steady. The coordinate system (xoy) is related to the coordinate system (x′ o′ y′) as follows:

\[
x' = x - V_o t, \quad y' = y, \quad t' = t.
\]

(4)
In this case the differentiation operators are replaced by the formulas
\[
\frac{\partial \varphi}{\partial x'} = \frac{\partial}{\partial x}, \quad \frac{\partial \psi}{\partial y'} = \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial x} \frac{\partial}{\partial t} = V_0 \frac{\partial}{\partial x}.
\]

Since the motion under consideration is steady, from (1) it follows that the derivatives of the potentials are related by the following equations of motion:
\[
(M_1^2 - 1) \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial y^2}, \quad (M_2^2 - 1) \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial y^2}, \quad M_1 = \frac{V_0}{a}, \quad M_2 = \frac{V_0}{b}.
\]

Hence, the components of the vector \( \mathbf{V} \) and the components of the stress tensor (3) can be represented as
\[
V_x = \frac{\partial u_x}{\partial t} = V_0 \left( \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \right), \quad V_y = \frac{\partial u_y}{\partial t} = V_0 \left( \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \right),
\]
\[
\frac{\sigma_{xx}}{\mu} = \left( 2 + M_2^2 - 2M_1^2 \right) \frac{\partial^2 \varphi}{\partial x^2} + 2 \frac{\partial^2 \psi}{\partial x \partial y},
\]
\[
\frac{\sigma_{yy}}{\mu} = \left( -2 - M_2^2 \right) \frac{\partial^2 \varphi}{\partial x^2} - 2 \frac{\partial^2 \psi}{\partial x \partial y},
\]
\[
\frac{\sigma_{xy}}{\mu} = 2 \frac{\partial^2 \psi}{\partial x \partial y} - \left( 2 - M_2^2 \right) \frac{\partial^2 \psi}{\partial x^2}.
\]

Let us consider the motion of the distributed load \( q_x/\mu = -q(x) \) and \( q_y/\mu = -p(x) \) at a velocity such that this velocity is greater than the velocity of the transverse waves and is less than the velocity of the longitudinal waves: \( M_2 > 1 \) and \( M_1 < 1 \). Outside the contact region, the functions \( q(x) \) and \( p(x) \) are equal to zero. Without loss of generality, we assume that the corresponding coordinate system can be chosen in such a way that the contact region takes the form \( y = 0, \ |x| < L \). In this region the functions \( q(x) \) and \( p(x) \) are unknown; they should be determined from the boundary conditions given on the surface of the stamp. Up to a shift along the axis, the position of the stamp contour is unknown with respect to the contact region. Let the contour equation of the above rigid stamp be \( y = f(x - c) + \text{const} \). The boundary conditions are the velocity component \( V_y \) and the relation between the components of a force vector (as an example, we adopt the dry friction law with a friction coefficient \( k \)):
\[
y = 0, \quad |x| < L, \quad V_y = V_0 f'(x - c), \quad q(x) = -kp(x).
\]

The length \( L \) of the contact region and the shift \( c \) of the stamp contour with respect to this region can be found only during the solving procedure. The required solutions of the above equations are sought in the form
\[
\varphi = \text{Re} \Phi(x + i\alpha y), \quad \alpha = \sqrt{1 - M_1^2}; \quad \psi = \Psi(x - \beta y), \quad \beta = \sqrt{M_2^2 - 1}.
\]

On the boundary, by virtue of (4), the components of the stress vector can be written as
\[
\frac{\sigma_{yy}}{\mu} = -\left( 2 - M_2^2 \right) \text{Re} \Phi''(x + i\alpha y) + 2\beta \Psi''(x - \beta y),
\]
\[
\frac{\sigma_{xy}}{\mu} = -2\alpha \text{Im} \Phi''(x + i\alpha y) - \left( 2 - M_2^2 \right) \Psi''(x - \beta y).
\]

On the free surface outside the contact region, the stress vector components (6) are equal to zero:
\[
-\left( 2 - M_2^2 \right) \text{Re} \Phi''(x) + 2\beta \Psi''(x) = 0,
\]
\[
-2\alpha \text{Im} \Phi''(x) - \left( 2 - M_2^2 \right) \Psi''(x) = 0.
\]

These boundary conditions are fulfilled if
\[
\Psi''(x) = \frac{2 - M_2^2}{2\beta} \text{Re} \Phi''(x),
\]
\[
\left( 2 - M_2^2 \right)^2 \text{Re} \Phi''(x) + 4\alpha \beta \text{Im} \Phi''(x) = 0.
\]