Effective Constitutive Relations for Inelastic Composites

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Received December 14, 2012; in final form, September 4, 2013

Abstract—The first special boundary value problem in the mechanics of deformable solids is considered to derive the effective constitutive relations for a heterogeneous inelastic body. The problem is reduced to a number of auxiliary boundary value problems for functions dependent on the shape of the body and on the form of constitutive relations. In the case of a layer of nonuniform thickness, the problem of finding the effective constitutive relations is reduced to an operator equation whose solution is sought by an iterative method of successive approximations. An approximate analytical formula is proposed to find the effective constitutive relations for a laminated composite on the basis of known inelastic constitutive relations for its components. This approximate formula takes into account the character of structural anisotropy in a laminated composite and, in the elastic case, yields the exact values of the effective elastic modulus.

DOI: 10.3103/S0027133013060022

1. FORMULATION OF THE PROBLEM

Let a heterogeneous body occupy a volume \( V \) and be bounded by a surface \( \Sigma \). The body is at equilibrium under the action of distributed loads and displacements given on its surface. Let us introduce a Cartesian coordinate system. In the case of small strains, the stress-strain state of the body is described by the equations \([1]\)

\[
\sigma_{ij,j} = 0, \quad \sigma_{ij} = \tilde{F}_{ij}(x, \tilde{\varepsilon}), \quad \varepsilon_{kl} = \Delta_{klmn} u_{m,n} \quad (x \in V),
\]

(1)

where \( \sigma_{ij} \), \( \varepsilon_{ij} \), and \( u_i \) are the components of stresses, strains, and displacements and \( \tilde{F}_{ij}(x, \tilde{\varepsilon}) \) are the constitutive relations (here the check symbol indicates a time operator). Generally, the constitutive operators are nonlinearly dependent on the stress tensor components. The material constants and functions used in the constitutive relations depend on the coordinates.

On the boundary of the body, the following displacements are given:

\[
u_i|\Sigma = u_i^0 = \gamma_{ij} y_j, \quad y \in \Sigma, \quad \gamma_{ij} = \gamma_{ji} = \text{const}.
\]

(2)

The problem expressed by (1) and (2) is said to be the first special boundary value problem whose solution allows one to derive the corresponding effective constitutive relations for a heterogeneous body \([2]\).

It is easy to show that, in the case of this first boundary value problem, we have

\[
<\varepsilon_{ij}> = \frac{1}{V} \int_V \varepsilon_{ij} dV = \gamma_{ij}.
\]

Solving this problem, we determine the displacements \( u_i(x, \gamma) \), the strains \( \varepsilon_{ij}(x, \gamma) \), and the stresses \( \sigma_{ij}(x, \gamma) \). Using a volume-averaging procedure, we get

\[
<\sigma_{ij}> = \frac{1}{V} \int_V \tilde{F}_{ij}(x, \varepsilon(x, \gamma)) dV = \tilde{h}_{ij}(\gamma) = \tilde{h}_{ij}(<\varepsilon>).
\]

(3)

The operator \( \tilde{h} \) is called the effective operator. The effective constitutive relations (3) are of the form \( <\sigma>\sim<\varepsilon> \) and allow one to express the volume-averaged stresses in terms of the volume-averaged strains.
The complexity of solving the first boundary value problem essentially depends on the form of the constitutive relations in use. This problem can be solved in a relatively simple manner in the case of a linearly elastic heterogeneous body when

\[ \tilde{F}_{ij}(x, \varepsilon) = C_{ijkl}(x)\varepsilon_{kl}. \]  

In the case of a linear viscoelastic material, the constitutive relations are of the form

\[ \tilde{F}_{ij}(x, \varepsilon) = \int_0^t \Gamma_{ijkl}(x, t, \tau)\varepsilon_{kl}(\tau) d\tau = C_{ijkl}(x)\varepsilon_{kl}(t) - \int_0^t \tilde{\Gamma}_{ijkl}(x, t, \tau)\varepsilon_{kl}(\tau) d\tau, \]

where \( \Gamma_{ijkl}(x, t, \tau) \) is the singular relaxation kernel and \( \tilde{\Gamma}_{ijkl}(x, t, \tau) \) is the regular relaxation kernel.

In the case of nonaging materials, the constitutive relations can be expressed in terms of the Stieltjes integrals [3]:

\[ \tilde{F}_{ij}(x, \varepsilon) = \int_0^t R_{ijkl}(x, t - \tau) d\varepsilon_{kl}(\tau) = C_{ijkl}(x)\varepsilon_{kl} + \int_0^t \frac{\partial R_{ijkl}(x, t - \tau)}{\partial t} \varepsilon_{kl}(\tau) d\tau. \]

Here \( C_{ijkl}(x) = R_{ijkl}(x, 0) \). In the case of small elastoplastic strains, we have

\[ \tilde{F}_{ij}(x, \varepsilon) = C_{ijkl}(x)\varepsilon_{kl} - C_{ij}(x, \varepsilon), \]

where \( C_{ij}(x, \varepsilon) \) are the second-rank tensor components dependent linearly on the strain tensor components. For a plastically incompressible isotropic material, in particular, we have

\[ C_{ij}(x, \varepsilon) = 2G(x)\omega(x, \varepsilon_u)D_{ijkl}\varepsilon_{kl}, \]

where \( G(x) \) is the elastic shear modulus, \( \varepsilon_u = \sqrt{2D_{ijkl}\varepsilon_{ij}\varepsilon_{kl}/3} \), \( \omega(x, \varepsilon_u) = 1 - \sigma_u(x, \varepsilon_u)/(3G\varepsilon_u) \) is the Hill’s plasticity function [4], and \( D_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2 - \delta_{ij}\delta_{kl}/3 \).

In our further discussion, we restrict ourselves to the constitutive relations of the form

\[ \sigma_{ij} = \tilde{F}_{ij}(x, \varepsilon) = C_{ijkl}(x)\varepsilon_{kl} - C_{ij}(x, \varepsilon). \]  

2. THE CASE OF A LINEARLY ELASTIC HETEROGENEOUS BODY

When the constitutive operator is of the form (4), the solution to problem (1), (2) can be represented as

\[ u_i(x, \gamma) = \gamma_{ij}x_j + N_{ikl}(x)\gamma_{kl}, \]

where \( N_{ikl}(x) \) are the sought continuous functions (they are symmetric with respect to the last two indices). Given the displacements, we can find the strains and, then, the stresses:

\[ \varepsilon_{ij} = (\Delta_{ijkl} + \Delta_{ijmn}N_{mkl,\gamma})\gamma_{kl}, \quad \sigma_{ij} = (C_{ijkl} + C_{ijmn}N_{mkl,\gamma})\gamma_{kl}. \]

Using the volume-averaged procedure for the stresses and taking into account that \( \gamma_{kl} = < \varepsilon_{kl} > \), we come to the following expressions for the effective elastic moduli in terms of the \( N_{ikl} \) functions:

\[ h_{ijkl} = (C_{ijkl} + C_{ijmn}N_{mkln,\gamma}). \]

Using the equilibrium equations (1) and the boundary conditions (2) and assuming that the values of \( \gamma_{ij} \) are arbitrary, we obtain the following equations and boundary conditions for the \( N_{ikl} \) functions:

\[ (C_{ijkl} + C_{ijmn}N_{mkln,\gamma})_{,j} = 0, \quad N_{mkln,\gamma} = 0. \]

It is not difficult to show that the effective elastic moduli obtained by (7) from the solution to the boundary value problem (8) are symmetric and positive definite:

\[ h_{ijkl} = h_{ijlk} = h_{klji} = h_{klij}, \quad h_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > m\varepsilon_{ij}\varepsilon_{ij}, \quad m > 0 \quad \forall \varepsilon_{ij} = \varepsilon_{ji} \neq 0. \]

The proof of this fact is similar to that given in [3] for a composite with a regular structure.