Deflected Mode of a Strip of Tapered Thickness Weakened by a Crack with End Yielded Regions

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Abstract—The paper considers a problem of fracture mechanics for a strip weakened by an end-to-end straightline crack with end yielded regions. It is believed that the strip of tapered thickness with an arbitrarily located crack is subjected to force loading. Influence of variability of strip thickness and plastic deformation on crack growth is studied.

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Strips of tapered thickness are widely used in engineering because it is often needed to give a product the desired properties by a change of its thickness. Let us consider a homogeneous isotropic strip of tapered thickness weakened by an end-to-end straightline crack. Let us note through the width and thickness of the strip, and through $x$, $y$ are, respectively, the smallest and the largest values of the thickness of the strip.

Let the strip with an arbitrarily located crack be subjected to force loading. We believe that the stresses on the boundary $y = \pm L$ of the strip are set (conditions of the first fundamental problem). The material of the strip is considered to be elastically perfectly plastic, satisfying the conditions of plasticity of the Tresca–St Venant criterion. When considering the problem, we use a rectangular Cartesian coordinate system $xOy$ assuming that the Cartesian coordinates $xy$ in the middle plane of the strip are a plane of symmetry. Under the action of stresses caused by the power load on the boundary $y = \pm L$ of the strip, plastic deformation regions will occur at the tips of the crack. Investigation of the stress state in this strip in the elastic formulation has shown [1] that the first regions of plastic deformation are expected at the continuation of the crack line.

On the basis of a calculation model of Leonov, Panasyuk, and Dugdale, a plastic region will be a thin layer at the continuation of the crack. As numerous experiments have shown [2], the plastic regions will present in these cases segments of length $d_1 = l_1 - \lambda_1$ and $d_2 = l_1 - \lambda_2$, located on the extension of the crack. We believe that the thickness of the strip $2h(x, y)$ satisfies the conditions $0 < h_1 \leq h(x, y) \leq h_2$, where $h_1$ and $h_2$ are, respectively, the smallest and the largest values of the thickness of the strip.

The function of the strip thickness can be expressed as [1]

$$h(x, y) = h_0[1 + \varepsilon \bar{h}(x, y)],$$

where $h_0 = (h_1 + h_2)/2$; $\varepsilon = (h_2 - h_1)/(h_2 + h_1)$ is a small parameter; and $\bar{h}(x, y)$ a known dimensionless variable function $(-1 \leq \bar{h}(x, y) \leq 1)$. For a given law of variation of the thickness, $\varepsilon$ is constant.

The edges of the crack are free from external loads. The boundary conditions at its edges have the following form

$$\sigma_y - i\tau_{xy} = p_1(x) \quad \text{at} \quad y = L, \quad -\infty < x < \infty;$$

$$\sigma_y - i\tau_{xy} = p_2(x) \quad \text{at} \quad y = -L, \quad -\infty < x < \infty,$$

where $p_1(x)$ and $p_2(x)$ are functions decreasing at infinity and satisfying the conditions of statics.

We search for the solution of the system of equations of statistical deformation of the strip of tapered thickness in the form of expansions by a small parameter [1]. In the resulting equations, the equation of the zeroth approximation coincides with the equations of a classical plane problem of elasticity theory, and the equations of the first approximation, and suc-
cessive approximations, coincide with the same differential equations with the bulk force determined from the solution of the previous approximations. To solve the problem, in each approximation we use the superposition principle. In each approximation there is the deflected mode of a strip with a crack in the plastic regions that can be represented as the sum of two deflected modes. The first state will be determined by solving the plane problem of elasticity theory for the strip in the absence of cracks. The second state will be determined by solving problems for the plastic strip with the plastic crack in the plastic end regions, on which banks there are forces of equal magnitude and opposite sign determined by the first state of stress.

Let us consider the zeroth approximation. The first state for the zeroth approximation (for a strip of constant thickness) in the absence of cracks is known (for example, [3]). The boundary conditions of the problem, determining the second state in the zeroth approximation, take the form

\[
\sigma_y^{(0)} = 0, \quad \tau_{xy}^{(0)} = 0 \quad \text{at} \quad y = L; \quad \sigma_y^{(0)} = 0, \quad \tau_{xy}^{(0)} = 0 \quad \text{at} \quad y = -L, \tag{1}
\]

at \( y_1 = 0 \)

\[
\sigma_{y_1}^{(0)} = -p_x^{(0)}(x_1), \quad \tau_{x_1y_1}^{(0)} = -p_{a_1}^{(0)}(x_1) \quad \text{outside end regions;}
\]

\[
\sigma_{y_1}^{(0)} = \sigma_x - p_x^{(0)}(x_1), \quad \tau_{x_1y_1}^{(0)} = \tau_x - p_{a_1}^{(0)}(x_1) \quad \text{on banks of end zones of cracks.} \tag{2}
\]

Here \( p_x^{(0)}(x_1) \) and \( p_{a_1}^{(0)}(x_1) \) are the normal and shear stresses occurring in the continuous strip along the axis \( x_1 (y_1 = 0) \) from the effect of stresses on the boundary \( y = \pm L \) of the strip in the zeroth approximation.

The boundary conditions (1) and (2), with the help of the Kolosov–Muskeshilvili formulas [4], one can write as a boundary-value problem for finding two analytic functions \( \Phi^{(0)}(z) \) and \( \Psi^{(0)}(z) \):

\[
\Phi^{(0)}(z) + \Phi^{(1)}(z) + z\Phi^{(0)}(z) + \Psi^{(0)}(z) = 0 \quad \text{at} \quad y = L, \tag{3}
\]

\[
\Phi^{(0)}(z) + \Phi^{(1)}(z) + z\Phi^{(0)}(z) + \Psi^{(0)}(z) = 0 \quad \text{at} \quad y = -L,
\]

\[
\Phi^{(0)}(t) + \Phi^{(1)}(t) + r\Phi^{(0)}(t) + \Psi^{(0)}(t) = f^{(0)}(t) \quad \text{at} \quad y_1 = 0,
\]

where

\[
f^{(0)}(t) = \begin{cases} 
-(p_x^{(0)}(t) - i p_{a_1}^{(0)}(t)) & \text{on} \quad L', \\
\sigma_x - i \tau_x - (p_x^{(0)}(t) - i p_{a_1}^{(0)}(t)) & \text{on} \quad L'';
\end{cases}
\]

\( t \) is an affix of the bank points of the crack with end regions; \( L' \) is a segment out of the endpoints of the crack regions; \( L'' \) is a collection of end regions of the crack.

Complex potentials \( \Phi^{(0)}(z) \) and \( \Psi^{(0)}(z) \) are sought in the form of [3]

\[
\Phi^{(0)}(z) = \frac{1}{2\pi i} \sum_{k=0}^{2} \int_{L_k} \frac{g_x(t) dt}{t - z_k}, \tag{4}
\]

\[
\Psi^{(0)}(z) = \frac{1}{2\pi} \sum_{k=0}^{2} e^{-2i\alpha_k} \int_{-i\alpha_k}^{0} \left[ \frac{g_x(t)}{t - z_k} - \frac{g_x(t)}{(t - z_k)^2} T_k e^{i\alpha_k} \right] dt
\]

where \( T_k = te^{i\alpha_k} + z_k^0, z_k = e^{-i\alpha_k}(z - z_k^0) \).

Satisfying the boundary conditions (3), after some transformations we obtain a system of three integral equations in relation to \( g_k^0 \) \((k = 0, 1, 2)\). In the system of three singular integral equations, we eliminate