Special Solutions of the Chapman—Kolmogorov Equation for Multidimensional-State Markov Processes with Continuous Time

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Received May 12, 2015

Abstract—The bilinear Chapman—Kolmogorov equation determines the dynamical behavior of Markov processes. The task to solve it directly (i.e., without linearizations) was posed by Bernstein in 1932 and was partially solved by Sarmanov in 1961 (solutions are represented by bilinear series). In 2007–2010, the author found several special solutions (represented both by Sarmanov-type series and by integrals) under the assumption that the state space of the Markov process is one-dimensional. In the present paper, three special solutions have been found (in the integral form) for the multidimensional-state Markov process. Results have been illustrated using five examples, including an example that shows that the original equation has solutions without a probabilistic interpretation.

Keywords: multidimensional-state Markov process with continuous time, solutions of the Chapman—Kolmogorov equation, transition probability, functional equations.

DOI: 10.3103/S1063454116020114

The bilinear integral Chapman—Kolmogorov equation determines the probability of the transition of a trajectory of a Markov stochastic process from the initial state (at an arbitrary time) to a neighborhood of any accessible state (at the next time). An immense number of publications are devoted to methods to solve that equation (see, e.g., the encyclopedia [1]), but direct methods (i.e., methods that do not use linear differential or integrodifferential equations) are not mentioned even in [1]. The problem to solve the said equation directly was posed by Bernstein in 1932 (see [2, p. 247]), but the first special solution was only obtained by Sarmanov in 1961 (see [3]): for stationary transition-probability densities, special solutions are represented by bilinear series with respect to eigenfunctions of the kernel of the operator generated by the Markov process. In [4, 5], the Chapman—Kolmogorov equation is reduced to functional equations; once they are solved, we obtain other representations of the transition-probability density both by bilinear series of the Sarmanov type and by integrals. In [4, 5], the only essential restriction is that the state space of the Markov process is assumed to be one-dimensional. In the present paper, that restriction is partially taken off. Also, the used method is applied to compute multiple integrals of a special kind occurring in problems related to ejections of Gauss processes (see [5]).

Introduce the notation

\[ \alpha = (\alpha_1, \ldots, \alpha_n) \in \Omega \subset \mathbb{R}_n, \quad d\alpha = d\alpha_1 \ldots d\alpha_n, \quad \alpha_k \in \mathbb{R}_1, \quad k = 1, \ldots, n, \]

where \( \alpha \) is assumed to be any of \( n \)-dimensional vectors \( \lambda, x, y, \) and \( z \) that occur below.

Consider a Markov process with an \( n \)-dimensional state space and continuous time. It is known from [1], it is known that such a process is completely characterized by its initial distribution (at a time \( s \)) and the probability density (it might be a generalized function as well) for the transition \( \pi_{s \to t} (x \to y) \) of the process trajectory from a point \( x \) (at the time \( s \)) to a neighborhood of a point \( y \) (at a time \( t \)) found from the Chapman—Kolmogorov equation

\[ \pi_{s \to t} (x \to z) = \int_\Omega \pi_{s \to \tau} (x \to y) \pi_{\tau \to t} (y \to z) dy, \quad (1) \]

where \( x, y, z \in \Omega \subset \mathbb{R}_n, s \leq \tau \leq t. \)
We look for a solution of Eq. (1) in the form
\[
\pi_{x\to y}(x \to y) = \int_{\Lambda} \psi_{st}(\lambda) \prod_{k=1}^{n} u_{\lambda_k}(y_k) v_{f_{st}^{(k)}(\lambda_k)}(x_k) d\lambda,
\]
where \( \Lambda \subseteq \mathbb{R}_n, \lambda_k, x_k, y_k \in \mathbb{R}_1, \psi_{st}(\lambda) \) is a continuous real function of the vector \( \lambda \), \( f_{st}^{(k)}(\lambda_k) \) are real functions, and \( u_{\lambda_k}(y_k) \) and \( v_{\lambda_k}(x_k) \) are real functions such that
\[
\int_{\Omega_k} u_{\lambda_k}(x_k) v_{\mu_k}(x_k) dx_k = \delta(\lambda_k - \mu_k), \quad k = 1, \ldots, n.
\]
In (3), \( \delta(\lambda_k) \) is the delta-function, \( \lambda_k \in \Lambda_k \subseteq \mathbb{R}_1, \mu_k \in \Lambda_k \subseteq \mathbb{R}_1, \Omega = \prod_{k=1}^{n} \Omega_k \) and \( \Lambda = \prod_{k=1}^{n} \Lambda_k \). Such function pairs \( u_{\lambda_k}(x_k) \) and \( v_{\lambda_k}(x_k) \) are used in the theory of Fourier, Hankel, and Kontorovich–Lebedev integral transformations (see [4]). For example, for \( \Omega_k = [0, \infty), \Lambda_k = (-\infty, \infty) \), the Fourier cosine-transformation satisfies the relation (3) if
\[
\pi_{x\to y}(x \to y) = \int_{\Lambda} \psi_{st}(\lambda) \prod_{k=1}^{n} u_{\lambda_k}(y_k) v_{\lambda_k}(x_k) dx_k = \pi_{y\to x}(y \to x),
\]
and it is treated as a kernel of a linear functional over continuous functions of the variable \( \lambda_k \). In the sequel, it is assumed that \( \Omega_k \) and \( \Lambda_k \) do not depend on variables of Eqs. (1) and (2), the integrand functions in (1) and (2) do not depend on \( \Omega_k \) and \( \Lambda_k \), and \( \psi_{st}(\lambda) \) is continuous with respect to \( \lambda_k \in \Lambda_k \). Those assumptions are imposed to justify the change of the integrating order (at least, for the case of the Fourier transformations used in the examples given below).

Substituting (2) in (1), changing the order of the integrating (with respect to \( y_k \) and to \( \lambda_k \)) at the right-hand part of (1) and using (3), we obtain the equation
\[
\pi_{x\to y}(x \to z) = \int_{\Lambda} \psi_{st}(f_{tr}(\lambda)) \psi_{tr}(\lambda) \prod_{k=1}^{n} u_{\lambda_k}(z_k) v_{f_{tr}^{(k)}(\lambda_k)}(x_k) d\lambda,
\]
which has a same type as (2). It is satisfied for
\[
\psi_{st}(\lambda) \prod_{k=1}^{n} v_{f_{tr}^{(k)}(\lambda_k)}(x_k) = \psi_{st}(f_{tr}(\lambda)) \prod_{k=1}^{n} v_{f_{tr}^{(k)}(\lambda_k)}(x_k),
\]
where
\[
f_{st}(\lambda) = \{f_{st}^{(1)}(\lambda_1), \ldots, f_{st}^{(n)}(\lambda_n)\}.
\]
To solve Eq. (4), introduce the following auxiliary assumptions.

First, we select functions \( f_{st}^{(k)}(\lambda_k) \) such that
\[
f_{st}^{(k)}(f_{tr}^{(k)}(\lambda_k)) = f_{st}^{(k)}(\lambda_k), \quad k = 1, 2, \ldots, n.
\]
Then, Eq. (4) is reduced to the following simpler form:
\[
\psi_{st}(\lambda) = \psi_{st}(f_{tr}(\lambda)) \psi_{tr}(\lambda).
\]
For \( \tau = t \), from (5)–(6), we conclude that
\[
f_{st}^{(k)}(\lambda_k) = \lambda_k, \quad \psi_{tr}(\lambda) = 1, \quad k = 1, 2, \ldots, n.
\]
Thus, Eq. (4) is decomposed into two functional equations (Eq. (5) and Eq. (6)). First, we have to find \( f_{st}^{(k)}(\lambda_k) \) from (5); then, using the found, we have to find \( \psi_{tr}(\lambda) \) from (6).

Let
\[
f_{st}(\lambda) = A(s, t)\lambda = \{A_1(s, t)\lambda_1, A_2(s, t)\lambda_2, \ldots, A_n(s, t)\lambda_n\},
\]
i.e., \( f_{st}^{(k)}(\lambda_k) = A_k(s, t)\lambda_k \). Then, from (5), we obtain that
\[
A_k(s, t)A_k(\tau, t) = A_k(s, t), \quad k = 1, 2, \ldots, n.
\]