Regularization in the Mosolov and Myasnikov Problem with Boundary Friction

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Abstract—We propose an iterative algorithm for solving a semicoercive nonsmooth variational inequality. The algorithm is based on the stepwise partial smoothing of the minimized functional and an iterative proximal regularization method. We obtain a solution to the variational Mosolov and Myasnikov problem with boundary friction as a limit point of a sequence of solutions to stable auxiliary problems.

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1. PROBLEM DEFINITION

Let \( \Omega \subset \mathbb{R}^2 \) be a finite domain with a sufficiently smooth boundary \( \Gamma \). Consider the problem

\[
J(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, d\Omega - \int_{\Omega} fv \, d\Omega + \int_{\Omega} g_1 |\nabla v| \, d\Omega + \int_{\Gamma} g_2 |v| \, d\Gamma \rightarrow \min, \quad v \in W^{1,2}_2(\Omega),
\]

(1)

where \( g_1 > 0, g_2 > 0, g_1, g_2 \) is a constant, \( f \in L^2(\Omega) \).

The functional \( J(v) \) is nondifferentiable and not strongly convex in the space \( W^{1,2}_2(\Omega) \). In [1] one studies a partially smoothed problem in the form

\[
J_\varepsilon(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, d\Omega - \int_{\Omega} fv \, d\Omega + \int_{\Omega} g_1 \sqrt{|\nabla v|^2 + \varepsilon^2} \, d\Omega + \int_{\Gamma} g_2 |v| \, d\Gamma \rightarrow \min, \quad v \in W^{1,2}_2(\Omega),
\]

(2)

where \( \varepsilon \) is a sufficiently small positive parameter.

In contrast to the classical Mosolov and Myasnikov problem with the adhesion boundary condition [2], the uniqueness theorem is not proved for problem (1), because the quadratic form \( \int_{\Omega} |\nabla v|^2 \, d\Omega \) is only a seminorm in the space \( W^{1,2}_2(\Omega) \).

The uniqueness theorem for problem (2) in the space \( W^{2,2}_2(\Omega) \) is proved in [1].

Problem (2) is equivalent to the variational inequality

\[
u \in W^{1,2}_2(\Omega) : \int_{\Omega} \left( \nabla u \nabla (v-u) + g_1 \frac{\nabla u \nabla (v-u)}{\sqrt{|\nabla u|^2 + \varepsilon^2}} - f(v-u) \right) \, d\Omega \\
+ \int_{\Gamma} g_2 \left( |v| - |u| \right) \, d\Gamma \geq 0 \quad \forall v \in W^{1,2}_2(\Omega).
\]
The solvability condition for problems (1) and (2) looks as follows:

\[ \int_{\Gamma} g_2 \, d\Gamma - \left| \int_{\Omega} f \, d\Omega \right| > 0 \]

2. THE METHOD OF ITERATIVE PROXIMAL REGULARIZATION WITH PARTIAL SMOOTHING OF THE MINIMIZED FUNCTIONAL

The iterative proximal regularization method for semismoothed problem (2) with a fixed small parameter \( \varepsilon \) is studied in [1].

In this paper we show that the concurrent application of the iterative proximal regularization method and the stepwise smoothing of the term \( \int_{\Omega} g_1 \sqrt{|\nabla v|} \, d\Omega \) in the initial functional allows us to construct a sequence of elements that converges to one of solutions of problem (1).

A similar approach is considered in [3] for the classical Mosolov and Myasnikov problem, where the minimized functional is strongly convex in the space \( W^{1,2}_0(\Omega) \).

The solution algorithm for problem (1) looks as follows:

a) set two sequences of positive numbers \( \delta_k, \varepsilon_k \), \( k = 1, 2, \ldots \), and a starting point \( u^0 \in W^{1,2}_0(\Omega) \);

b) on the \( k \)th iteration we define the point \( u^k \) from the condition \( \| u^k - \bar{u}^k \|_{W^{1,2}_0(\Omega)} \leq \delta_k \), where

\[
\bar{u}^k = \arg \min_{v \in W^{1,2}_0(\Omega)} \left\{ J_{\varepsilon_k}(v) + \frac{1}{2} \| v - u^{k-1} \|^2_{L_2(\Omega)} \right\},
\]

\[
J_{\varepsilon_k}(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, d\Omega - \int_{\Omega} f v \, d\Omega + \int_{\Omega} g_1 \sqrt{|\nabla v|^2 + \varepsilon_k^2} \, d\Omega + \int_{\Gamma} g_2 |v| \, d\Gamma, \quad k = 1, 2, \ldots \quad (3)
\]

The regularizing term \( 1/2 \| v - u^{k-1} \|^2_{L_2(\Omega)} \) in the functional \( J_{\varepsilon_k}(v) \) provides the strong convexity of the minimized functional. This guarantees the existence and uniqueness of the element \( \bar{u}^k \) with any fixed \( k \) and allows us to use efficient optimization methods for finding an approximate solution \( u^k \).

For problem (1) we consider an auxiliary problem with the regularization of the original functional

\[
J(v) + \frac{1}{2} \| v - u^{k-1} \|^2_{L_2(\Omega)} \rightarrow \min, \quad v \in W^{1,2}_0(\Omega).
\]

Let \( \bar{u}^k \) stand for the unique solution of problem (4). The corresponding variational inequality takes the form

\[
\bar{u}^k \in W^{1,2}_0(\Omega) : \int_{\Omega} \nabla \bar{u}^k \nabla (v - \bar{u}^k) \, d\Omega + \int_{\Omega} \bar{u}^k (v - \bar{u}^k) \, d\Omega \]

\[
- \int_{\Omega} (f + u^{k-1}) (v - \bar{u}^k) \, d\Omega + g_1 \int_{\Omega} |\nabla v| \, d\Omega - g_1 \int_{\Omega} |\nabla \bar{u}^k| \, d\Omega + g_2 \int_{\Gamma} |v| \, d\Gamma - g_2 \int_{\Gamma} |\bar{u}^k| \, d\Gamma \geq 0 \quad \forall v \in W^{1,2}_0(\Omega).
\]

Theorem 1. Let the condition

\[
\sum_{k=1}^{\infty} (\delta_k + \varepsilon_k^{1/2}) < +\infty
\]

be fulfilled. Then the sequence \( \{u^k\} \) converges to one of solutions to problem (1) in the norm of the space \( W^{1,2}_0(\Omega) \).