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On Algebras Over Multicategories

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Abstract—We introduce a notion of a monoidal category over verbal category. In such categories we define algebras over multicategories over the same verbal categories. We also explicitly compute categories of algebras for two classes of multicategories.

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INTRODUCTION

Here we continue the study initiated in [1–4]. The main object of consideration is multigraded (color) operads, also called multicategories. The author has already introduced the sweeping generalization of multicategories, namely, multicategories over verbal categories. In order to study the algebra categories over these multicategories in more general than set-theoretical cases we introduce one new class of monoidal (tensor) categories, i.e., a monoidal categories over a verbal algebras. This allows us to define a notion of algebra over a multicategory over a verbal category as the object of a monoidal category.

The paper consists of three Sections. The first Section introduces the notion of the monoidal category over the verbal category. We also point out certain links with the double category theory and give a number of examples. In the second Section we show that the known functor that maps a monoidal category over a verbal category into multicategory in fact constructs a multicategory over the relative verbal category. Then we introduce the notion of an algebra over a multicategory over a verbal category as a multifunctor from the given multicategory into the already mentioned multicategory constructed by the monoidal category. Finally, in the third Section we apply the newly-built notation for the explicit calculation of the algebra categories over the two naturally defined multicategory classes (matrix and semigroup algebra operads generalizations of [5]).

The notation follows the authors works [1, 3, 4].

1. MONOIDAL CATEGORIES OVER VERBAL CATEGORIES

We start with the lemma, whose formulation in compressed form (in double categories terms [6]) actually contain multicategories over verbal categories definition. Throughout the paper the top line denotes finite ordered index sequence, most often the numbered letter set, for example, $\overline{x} = x_1x_2\dots x_n$. The other names of such sequences are strings or words in some alphabet. Recall also the notion $[n] = \{0, 1, \dots, n\}$. Verbal categories then are categories with objects of type $[n]$. The morphisms of these categories are mappings of the form $[n] \to [m]$, under which only zero and no other element maps into zero. All the mappings with such a property are morphisms of the verbal category FSet. The other important example of a verbal category is the category Σ whose morphisms are the bijections of type $[n] \to [n]$ (of the same property with respect to zero). Moreover, $\Sigma([n], [n]) = \Sigma_n$ is the substitution group of degree n. The exact definition of the verbal categories can be found in [2] and [4]. Further on we recall certain properties of the verbal categories.

After we compare the multicategories over verbal categories ([1], definition 3) and double categories definitions we obtain

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Lemma 1. *Let R be a multicategory over the verbal category W ([1, 4]). Then there exists the double category $\mathcal{D}R_W$ whose objects have the following description: The objects of $\mathcal{D}R_W$ are finite ordered sequences (strings) of objects from R. The horizontal arrows $\overline{x} = x_1 \dots x_n \to y_1 \dots y_m$ are arrows $\omega_1 \dots \omega_m$, here $\omega_i : \overline{x}_i \to y_i$ are multiarrows from R and $\overline{x}_1 \dots \overline{x}_m = \overline{x}$. The vertical arrows have the form $f : y_1 \dots y_m \to y_{f(1)} \dots y_{f(k)}$, here $f \in W^{\mathrm{op}}([k],[m])$ is the morphism of the category dual to W (it is represented by the mapping from $[k]$ to $[m]$). The squares $\mathcal{D}R_W$ by definition then are of the type*

$$\begin{array}{ccc} \overline{x}_{f(1)} \dots \overline{x}_{f(k)} & \xrightarrow{\omega_{f(1)} \cdots \omega_{f(k)}} & y_{f(1)} \dots y_{f(k)} \\ {\scriptstyle f^*\alpha}\uparrow & & {\scriptstyle f}\uparrow \\ \overline{x}_1 \dots \overline{x}_m & \xrightarrow{\omega_1 \dots \omega_m} & y_1 \dots y_m, \end{array}$$

here $\alpha = \binom{\overline{x}_1 \dots \overline{x}_m}{y_1 \dots y_m}$.

We denote the category comprising the horizontal arrows of the double category $\mathcal{D}R_W$ by $\mathcal{C}R$. This is a strictly monoidal category, such that $\overline{x} \otimes \overline{y} = \overline{x}\,\overline{y}$. The morphisms have the similar behavior. In what follows we clarify the part of this category.

We now pass to the formulation of the paper main notion definition. The definition and basic properties of the monoidal categories can be found in [7] (pp. 188-191) and [8] (pp. 292-294). We denote the bifunctor "of the tensor product " which turns into the square sign in ([7], P. 188) by $\otimes$. It is known that each monoidal category is equivalent to strictly monoidal, i.e., to the one such that $(x \otimes y) \otimes z = x \otimes (y \otimes z)$. The proof of this fact can be found in [9] (theorem 1.2.15). So further on we study as a rule only strictly monoidal categories.

Let us then consider a strictly monoidal category K and assume that $S = \mathrm{Ob}(K)$. We define the category K^* whose objects are finite strings $\overline{x} = x_1 \dots x_n$, here $x_i \in S$, and morphisms from $\overline{x}$ to $\overline{y} = y_1 \dots y_m$ are the category K morphisms of the type $\omega : x_1 \otimes \cdots \otimes x_n \to y_1 \otimes \cdots \otimes y_m$. There exist naturally defined functors $K \to K^*$ (the object x maps into the string x of length 1) and $K^* \to K$ (the object-string $x_1 \dots x_n$ maps into the object $x_1 \otimes \cdots \otimes x_n$). It can be easily verified that this a category equivalence. In the definition that we introduce somewhat later in the text we use the notation of [4]. In particular, if W is a verbal category [2] and S is some symbol class, then W_S is a category whose objects are finite S symbol sequences and morphisms are of the type $f : \overline{s} = s_1 \dots s_n \to s_{f(1)} \dots s_{f(k)} = \overline{s}f$, here $f \in W^{\mathrm{op}}([k],[n])$. We say then [4] that this morphism is represented by the mapping f and simply denote it also by f. If we have two morphisms $f_i : [n_i] \to [m_i]$ $(i = 1, 2)$ of the verbal category W, then the mapping $f_1 \sqcup f_2 : [n_1 + n_2] \to [m_1 + m_2]$, that puts zero into zero, the elements j $(1 \le j \le n_1)$ into elements $f_1(j)$ and the elements $k + n_1$ into $f_2(k) + m_1$ (here $1 \le k \le n_2$) belongs to W by the verbal category definition. If we have two morphisms of W_S represented by f_1 and f_2 (the morphisms of W) that have the type $f_i : \overline{x}_i \to \overline{x}_i f_i$, then there exists a morphism $\overline{x}_1\overline{x}_2 \to (\overline{x}_1 f_1)(\overline{x}_2 f_2) = (\overline{x}_1\overline{x}_2)(f_1 \sqcup f_2)$ of the category W_S represented by the morphism $f_1 \sqcup f_2$ of W and denoted also by $f_1 \sqcup f_2$.

Definition 1. Consider a multisorted verbal category W_S (here W is the relative one-sorted verbal category) and a contravariant functor $\mathcal{F} : W_S \to K^*$ with the following properties:

1) the functor $\mathcal{F}$ is identical on the objects;

2) $\mathcal{F}(f \sqcup g) = \mathcal{F}(f) \otimes \mathcal{F}(g)$;

3) all diagrams of the type

$$\begin{array}{ccc} \overline{x}_{f(1)} \dots \overline{x}_{f(k)} & \xrightarrow{\omega_{f(1)} \otimes \cdots \otimes \omega_{f(k)}} & y_{f(1)} \dots y_{f(k)} \\ \uparrow{\scriptstyle \mathcal{F}(f^*\alpha)} & & \uparrow{\scriptstyle \mathcal{F}(f)} \\ \overline{x}_1 \dots \overline{x}_m & \xrightarrow{\omega_1 \otimes \cdots \otimes \omega_m} & y_1 \dots y_m \end{array} \qquad (1)$$

are commutative.

Here $\overline{x}_i$ stands for the string $x_{i,1} \dots x_{i,k_i}$, $f : [k] \to [m]$ is a morphism of W, $\alpha = \binom{\overline{x}_1 \dots \overline{x}_m}{y_1 \dots y_m}$.

Under these conditions we say that either K is a *strictly monoidal W-category* or we have a *W-structure* on K.