FUNCTIONAL ANALYSIS

On a Class of Differential Operators with Constant Coefficients
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Abstract—A linear differential operator $P(D) = P(D_1, \ldots, D_n)$ with constant coefficients is called almost hypoelliptic if all the derivatives $D^\alpha P$ of the characteristic polynomial $P(\xi_1, \ldots, \xi_n)$ can be estimated by $P$. The paper proves that if $P$ is an almost hypoelliptic operator and $f$ is an infinitely differentiable function, square-summable with a definite exponential weight, then any square summable with the same weight solution $u$ of the equation $P(D)u = f$ is again an infinitely differentiable function and $P(\xi) \to \infty$ as $\xi \to \infty$.

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1. NOTATION AND PROBLEM STATEMENT

Everywhere below, we assume that $N$ is the set of natural numbers, $N_0 = N \cup \{0\}$, $N^n_0 = N_0 \times \cdots \times N_0$ is the set of $n$-dimensional multiindices, $E^n$ and $R^n$ are $n$-dimensional, real Euclidean spaces with points $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ respectively and $C^n_0 = R^n \times iR^n$ ($i^2 = -1$).

Further, for any $\xi \in R^n$, $\zeta \in C^n$, $x \in E^n$ and $\alpha \in N^n_0$ we denote

$$|\zeta| = (|\zeta_1|^2 + \cdots + |\zeta_n|^2)^{1/2}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$$\xi^\alpha_\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}, \quad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n},$$

where either

$$D_j = \frac{\partial}{\partial \xi_j} \text{ or } D_j = \frac{1}{i} \frac{\partial}{\partial \xi_j}, \quad j = 1, \ldots, n.$$

We consider a linear differential operator

$$P(D) = \sum_{\alpha} \gamma_\alpha D^\alpha,$$  \hspace{1cm} (1.1)

where the sum is taken over a finite collection

$$(P) = \{\alpha \in N^n_0 : \gamma_\alpha \neq 0\},$$

and its complete symbol

$$P(\xi) = \sum_{\alpha \in (P)} \gamma_\alpha \xi^\alpha.$$  

The convex hull of the set $(P) \cup \{0\}$ is called characteristic polyhedron of the operator (or polynomial) $P$.

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For a linear differential operator (1.1), an integer \( m \geq 0 \) and any \( \delta \geq 0 \), we denote

\[
L_{2,\delta} \equiv \left\{ u : u \in L^2_{2,\delta}, \|u\|_{L_{2,\delta}} \equiv \left( \int |u(x)e^{-\delta|x|}|^2 \, dx \right)^{1/2} < +\infty \right\},
\]

\[
W^m_\delta \equiv \left\{ u : u \in L_{2,\delta}, \|u\|_{W^m_\delta} \equiv \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_{2,\delta}} < +\infty \right\},
\]

\[
N(P,\delta,m) \equiv \left\{ u : u \in L_{2,\delta}, \|u\|_{N(P,\delta,m)} \equiv \|P(D)u\|_{W^m_\delta} + \|u\|_{L_{2,\delta}} < +\infty \right\},
\]

\[
D(P) \equiv \{ \zeta : \zeta \in \mathbb{C}^n; P(\zeta) = 0 \}, \quad D(P,\delta) = \{ \zeta : \zeta \in D(P), |\text{Im} \zeta| < \delta \}.
\]

Besides, we set

\[
W^\infty_\delta \equiv \bigcap_{m=0}^{\infty} W^m_\delta \quad \text{and} \quad N(P,\delta,m) \equiv \bigcap_{m=0}^{\infty} N(P,\delta,m).
\]

In fact \( W^m_\delta(N(P,\delta,m)) \) becomes a Banach space and \( W^\infty_\delta(N(P,\delta)) \) becomes a Frechet space, given the seminorms

\[
\| \cdot \|_{W^m_\delta} \quad \text{and} \quad \|P(D) : \| \cdot \|_{W^m_\delta} + \| \cdot \|_{L_{2,\delta}}, \quad m = 0, 1, 2, \ldots.
\]

Clearly \( W^\infty_\delta \subset C^\infty, W^\infty_\delta \subset N(P,\delta) \) for any \( \delta > 0 \).

V. I. Burenkov in [1] proved that \( N(P,0) \subset W^\infty_0 \) if \( P(\xi) \neq 0 \) for \( \xi \in \mathbb{R}^n \) great enough. The present paper is mainly aimed at finding a condition on the symbol of the operator \( P(D) \), which ensures the existence of some \( \delta > 0 \) for which \( N(P,\delta) \subset W^\infty_\delta \).

**Definition 1.** (see Theorems 11.1.1 and 11.1.3 in [2]) A differential operator \( P(D) \) is called hypoelliptic if the following equivalent conditions are fulfilled:

(i) for any \( \alpha \in \mathbb{N}^n \), we have \( \frac{\partial^\alpha}{\partial \zeta^\alpha} P(\zeta) \equiv D^\alpha P(\zeta) \rightarrow 0 \) as \( |\zeta| \rightarrow \infty \).

(ii) \( d_P(\xi) \rightarrow \infty \) as \( |\xi| \rightarrow \infty \) \( (\xi \in \mathbb{R}^n) \),

where \( d_P(\xi) \) is the distance from the point \( \xi \) to the surface \( D(P) \).

Theorems 11.1.1 and 11.1.3 of [2] prove that a linear differential operator \( P(D) = P(D_1,\ldots,D_n) \) is hypoelliptic if and only if any generalized solution \( u \) of the equation \( P(D)u = 0 \) is an infinitely differentiable function.

**Definition 2.** (see [3]) The operator \( P(D) \) is called almost hypoelliptic if for any \( \nu \in \mathbb{N}^n_0 \) there exists a constant \( C_\nu > 0 \) such that

\[
|P^{(\nu)}(\xi)| \leq C_\nu (|P(\xi)| + 1), \quad \xi \in \mathbb{R}^n.
\]  

(1.2)

**Lemma 1.** (Lemma 11.1.4 in [2]) For any polynomial \( Q(\xi) \) of \( n \) variables, there exists a constant \( \mathcal{H} = \mathcal{H}(n, \text{ord} Q) > 0 \) such that

\[
\mathcal{H}^{-1} \leq d_Q(\xi) \sum_{\alpha \neq 0} \frac{|Q^\alpha(\xi)|^{1/|\alpha|}}{|Q(\xi)|} \leq \mathcal{H}
\]  

(1.3)

whenever \( \xi \in \mathbb{R}^n \) and \( Q(\xi) \neq 0 \).

It immediately follows from (1.3) that if

\[
|P(\xi)| \geq \epsilon, \quad |\xi| \geq C,
\]  

(1.4)

with some constants \( \epsilon > 0 \) and \( C > 0 \), then the polynomial \( P \) is almost hypoelliptic if and only if

\[
\rho_p \equiv \liminf_{t \rightarrow -\infty} \min_{|\xi| = t} d_P(\xi) > 0.
\]  

(1.5)

Note also that the polynomial \( P \) is hypoelliptic if and only if \( \rho_p = \infty \).