On Isometric Representations of the Semigroup $\mathbb{Z}_+\backslash\{1\}$

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Abstract—In this paper we study isometric representations of the semigroup $\mathbb{Z}_+\backslash\{1\}$. The notion of inverse representation is introduced and a complete (to within unitary equivalence) description of such representations of that semigroup is provided. A class of irreducible non-inverse representations ($\beta$-representations of the semigroup $\mathbb{Z}_+\backslash\{1\}$) is described.

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1. INTRODUCTION

In the paper [4] Coburn proved that all semiunitary representations of the semigroup of nonnegative integers by isometric operators generate canonically isomorphic algebras. Later a similar result for semigroups with archimedian order and total order have been proved by Douglas [5] and Murphy [7], respectively. In [1] was proved that all non-unitary isometric representations of a semigroup $S$ generate canonically isomorphic $C^*$-algebras if and only if the natural order on $S$ is total. A simple example of a semigroup with non-total order provides the semigroup $\mathbb{Z}_+\backslash\{1\}$, which originally was discussed by Murphy [7]. Later, Jang [6] pointed out two representations of this semigroup that generate canonically non-isomorphic $C^*$-algebras. Raeburn and Vittadello [9] studied all isometric representations of the semigroup $\mathbb{Z}_+\backslash\{1\}$ under certain condition.

The present paper is devoted to the isometric representations of the semigroup $\mathbb{Z}_+\backslash\{1\}$. Here the term isometric representation stands for a representation by isometric (more precisely, semiunitary) operators in a Hilbert space. We introduce a notion of inverse representation and show that there exist only two inverse irreducible representations (to within unitary equivalence), which are the same representations as in [6], [7], [9], [10]. We also study non-inverse representations of the semigroup $\mathbb{Z}_+\backslash\{1\}$.

2. INVERSE REPRESENTATIONS

Throughout the paper $S$ will stand for an abelian additive cancelative semigroup containing the neutral element 0 and not containing a group different from trivial. By $\Gamma$ we denote the Grothendieck group generated by the semigroup $S$. Recall that the group $\Gamma$ is a quotient of the semigroup $S \times S$ with respect to equivalence $(a, b) \sim (c, d)$ if and only if $a + d = b + c$, and the inverse of the quotient class $[(a, b)]$ is $[(b, a)]$. The notation $\Gamma = S - S$ is commonly used.

Let $T : S \rightarrow B(H_T)$ be the faithful non-unitary isometric representation of the semigroup $S$ into algebra $B(H_T)$ of all bounded linear operators on the Hilbert space $H_T$. Observe that in this case $T(0) = I$.

For any $a \in S$ by $T^*(a)$ we denote the conjugate of the operator $T(a)$. We have $T^*(a)T(a) = I$, where $I$ is the identity operator, and $T(a)T^*(a) = P_T(a)$, where $P_T(a)$ is the projection ($P_T(a) \neq I$).

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Let $T : S \to B(H_T)$ be an isometric representation of the semigroup $S$. An element $h_0 \in H_T$ we call *initial* for the operator $T(a)$ if $T^*(a)h_0 = 0$ for any $a \in S \setminus \{0\}$ and $||h_0|| = 1$.

The operators $T(a)$ and $T^*(b)$, $a, b \in S$, we call *trivial monomials*. A *monomial* is defined to be a finite product of trivial monomials. The set of all monomials forms a multiplicative involutive semigroup, which we will denote by $S^*_T$. On the semigroup $S$ we define the order: $a \prec b$ if $b = a + c$. Note that $S$ is a net with respect to this order.

**Lemma 2.1.** For any monomial $V$ there exist $x$ and $y$ in $S$ such that
\[
\lim_{c \in S} T^*(c)VT(c) = T^*(x)T(y),
\]
where $\lim_{c \in S}$ is the limit by the net $S$.

**Proof.** Let $V$ be a monomial represented as follows
\[
V = \prod_{i=1}^{n} T'(a_i),
\]
where $T'(a_i)$ is either $T(a_i)$ or $T^*(a_i)$. We split the set $\{a_i\}_{i=1}^{n}$ into two subsets: $\{a_{i_k}\}_{k=1}^{l}$ and $\{a_{i_j}\}_{j=l+1}^{n}$. The first subset consists of those $\{a_i\}_{i=1}^{n}$ for which the monomial $V$ involves the operator $T(a_i)$, while for the second $V$ involves the operator $T^*(a_i)$. Let
\[
a = \sum_{j=l+1}^{n} a_{i_j}, \quad b = \sum_{k=1}^{l} a_{i_k}.
\]
Using the equalities $T^*(s)T(s) = I$ and $T(s)T(t) = T(t)T(s)$ for any $s, t \in S$ we have
\[
T^*(c)VT(c) = T^*(a)T(b),
\]
where $c = a + b$. Therefore for any $d, c \prec d$
\[
T^*(d)VT(d) = T^*(a)T(b) = \lim_{c \in S} T^*(c)VT(c),
\]
and the result follows.

Observe that if $T^*(a)T(b) = T^*(c)T(d)$ for some $a, b, c$ and $d$ from $S$, then due to faithfulness of the representation we have $b + c = a + d$. This implies that to each monomial $V$ can be correspond a unique element $b - a$ from the Grothendieck group $\Gamma$. The element $b - a$ we call an *index* of the monomial $V$ and denote by $\text{ind}(V) = b - a$. This notion was introduced in [2] for regular representation of semigroup $S$.

**Lemma 2.2.** The following assertions hold:

1. The index of a monomial does not depend on its representation by elementary monomials.

2. The index of a product of monomials is equal to the sum of indices of factors:
\[
\text{ind}(V_1 \cdot V_2) = \text{ind}(V_1) + \text{ind}(V_2).
\]

Denote by $S^*_{0,T}$ the subsemigroup of the semigroup $S^*_T$ consisting of those $V$ for which $\text{ind}(V) = 0$.

An isometric representation $T : S \to B(H_T)$ is called *inverse representation* if $S^*_T$ is an inverse semigroup with respect to multiplication and involution, or equivalently, if $S^*_{0,T}$ is a semigroup of idempotents in $S^*_T$ (i.e., a semigroup of orthogonal projections). According to lemma 2.2 from [2] each semigroup $S$ possesses at least one inverse representation. On the other hand, as it was shown in [1], if the above defined order on $S$ is a total order, then all the isometric representations of $S$ are inverse.

A simple example of inverse representation is the representation $L : \mathbb{Z}_+ \to B(\ell^2(\mathbb{Z}_+))$ by the shift operator $L(n)e_m = e_{n+m}$, where $e_n(m) = \delta_{n,m}$ is the Kronecker symbol. Notice that the system $\{e_n(m)\}$ forms an orthonormal basis in $\ell^2(\mathbb{Z}_+)$, and in this case, the semigroup $\mathbb{Z}_+L$ is a bicyclic semigroup.