Fine Properties of Functions from Hajłasz–Sobolev Classes $M^p_\alpha$, $p > 0$, I. Lebesgue Points

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Abstract—Let $X$ be a metric measure space satisfying the doubling condition of order $\gamma > 0$. For a function $f \in L^p_{\text{loc}}(X)$, $p > 0$ and a ball $B \subset X$ by $I^p_B f$ we denote the best approximation by constants in the space $L^p(B)$. In this paper, for functions $f$ from Hajłasz–Sobolev classes $M^p_\alpha(X)$, $p > 0$, $\alpha > 0$, we investigate the size of the set $E$ of points for which the limit $\lim_{r \to 0} I^p_{B(x,r)} f = f^*(x)$ exists. We prove that the complement of the set $E$ has zero outer measure for some general class of outer measures (in particular, it has zero capacity). A sharp estimate of the Hausdorff dimension of this complement is given. Besides, it is shown that for $x \in E$

$$\lim_{r \to 0} \int_{B(x,r)} |f - f^*(x)|^q \, d\mu = 0, \quad 1/q = 1/p - \alpha/\gamma.$$  

Similar results are also proved for the sets where the "means" $I^p_{B(x,r)} f$ converge with a specified rate.

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1. INTRODUCTION

The classical Lebesgue theorem (see [1, Chapter 1, §1]) states that for any function $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, $p \geq 1$, almost all points are Lebesgue points, that is, for $\mu$-almost everywhere $x \in \mathbb{R}^n$ the limit

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu = f^*(x)$$  \hspace{1cm} (1.1)

exists and the function $f^*$ is equivalent to $f$, where $\mu$ is the Lebesgue measure on $\mathbb{R}^n$.

For more regular functions, the sizes of the Lebesgue exceptional set (that is, the set of points for which the limit (1.1) does not exist), can be estimated more precisely. For instance, for functions from Sobolev space $W^p_k(\mathbb{R}^n)$ with $1 < p < n/k$, this exceptional set has vanishing $W^p_k$-capacity, and its Hausdorff dimension is at most $n - kp$. Such type questions first were considered in [2], and then were generalized in [3–5]. A detailed history of similar results on $\mathbb{R}^n$ can be found in [6, Chapter 6.2].

Recently similar problems in more general situation, for Hajłasz–Sobolev classes $M^p_\alpha(X)$, $p \geq 1$, $\alpha > 0$, on metric spaces with measure (see definition below) have been studied by a number of authors (see [7–14], and references therein).

In the present paper, we also study estimates of the size of the Lebesgue exceptional set for functions from Hajłasz–Sobolev classes $M^p_\alpha(X)$, but for $p > 0$, in which case the functions from $M^p_0(X)$ can be non-summable. To this end, we first introduce a generalization of the notion of Lebesgue points, which does not use the integral averages over balls.

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2. THE BASIC NOTIONS AND NOTATION

Let \((X, d, \mu)\) be a metric space with a regular Borel measure \(\mu\) and a metric \(d\). We assume that the measure \(\mu\) satisfies the doubling condition, that is, there exists a number \(a_\mu > 0\), such that

\[
\mu(B(x, 2r)) \leq a_\mu \mu(B(x, r)), \quad x \in X, \quad r > 0.
\]

(2.1)

Note that the condition (2.1) can be given the following quantitative form: under (2.1) for some \(\gamma > 0\) (can be taken \(\gamma = \log_2 a_\mu\)) the following inequality holds:

\[
\mu(B(x, R)) \leq a_\mu \left(\frac{R}{r}\right) \gamma \mu(B(x, r)), \quad x \in X, \quad 0 < r \leq R.
\]

(2.2)

The constant \(\gamma\) plays the role of the dimension of the space \(X\).

For a ball \(B \subset X\), by \(r_B\) and \(x_B\) we denote the radius and center of \(B\), respectively, and by \(\lambda B\) we denote the concentric with \(B\) ball of radius \(\lambda r_B\). Throughout the paper we use the notation

\[
f_B = \int_B f \, d\mu = \frac{1}{\mu(B)} \int_B f \, d\mu
\]

for the average of a function \(f \in L^1_{\text{loc}}(X)\) over the ball \(B \subset X\).

By letter \(c\) will be denoted various positive constants, possibly depending on some parameters, but these dependence will not be essential for us. Besides, the notation \(A \lesssim B\) will mean \(A \leq cB\).

It is easy to check (see, e.g., [15, Lemma 3]) that for any number \(p > 0\), a ball \(B \subset X\) and a function \(f \in L^p(B)\) there exists a number \(I_B^{(p)} f\) to satisfy

\[
\left(\int_B |f(y) - I_B^{(p)} f|^p \, d\mu(y) \right)^{1/p} = \inf \left(\int_B |f(y) - I_B^{(p)} f|^p \, d\mu(y) \right)^{1/p}.
\]

Note that the number \(I_B^{(p)} f\) generally is not uniquely defined. In this case we take any of the possible values of \(I_B^{(p)} f\). The numbers \(I_B^{(p)} f\) we play the role of integral averages \(f_B\) for nonsummable functions.

For a function \(f \in L^p(X)\) by \(D_\alpha[f]\) we denote the class of all nonnegative \(\mu\)-measurable functions \(g\), for each of which there exists a set \(E \subset X\) with \(\mu(E) = 0\), such that

\[
|f(x) - f(y)| \leq [d(x, y)]^\alpha [g(x) + g(y)], \quad x, y \in X \setminus E.
\]

The elements of \(D_\alpha[f]\) are called generalized \(\alpha\)-gradients of function \(f\). We list some simple properties of the generalized \(\alpha\)-gradients \(D_\alpha[\cdot]\) that will be used below without additional comments and references:

\[
D_\alpha[f] \subset D_\alpha[|f|], \quad D_\alpha[f + \text{const}] = D_\alpha[f],
\]

\[
g_f \in D_\alpha[f] \text{ and } g_v \in D_\alpha[v] \implies g_f + g_v \in D_\alpha[f + v],
\]

\[
g_f \in D_\alpha[f] \implies cg_f \in D_\alpha[cf], \quad c > 0,
\]

if \(g_i \in D_\alpha[f_i]\) and \(\sup_i f_i < +\infty\) \(\mu\)-almost everywhere, then

\[
\sup_{i \in \mathbb{N}} g_i \in D_\alpha \left( \sup_{i \in \mathbb{N}} f_i \right).
\]

(2.3)

We introduce the scale of Hajłasz–Sobolev classes:

\[
M^p_\alpha(X) = \{ f \in L^p(X) : D_\alpha[f] \cap L^p(X) \neq \emptyset \}, \quad 0 < p < \infty, \quad \alpha > 0.
\]

These spaces are normed as follows:

\[
\|f\|_{M^p_\alpha(X)} = \|f\|_{L^p(X)} + \inf \{ \|g\|_{L^p(X)} : g \in D_\alpha[f] \cap L^p(X) \}
\]

(2.4)

(note that for \(0 < p < 1\) the expression (2.4) is only a pre-norm). For \(\alpha = 1\) these spaces were introduced in the paper by P. Hajłasz [7], where it was shown that \(M^1_1(\mathbb{R}^n)\) coincides with the classical Sobolev