Nonuniform Parametric Sound Generation in Acoustic Resonator

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Abstract—A theory of sound generation in an open acoustic resonator excited by a monochromatic sound beam, under nonuniform modulation of the speed of sound in the resonator medium, has been developed. The spectral distribution and temporal profile of the resonator stationary radiation are calculated.

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Recently much interest has been shown in the formation of sound beams with a complex and controlled temporal structure by parametric conversion of acoustic fields, based on modulation the speed of sound. For example, a broadband modulation of the speed of sound makes it possible to efficiently compress sound pulses [1]. Parametric generation of acoustic fields in a resonator under certain conditions allows one to obtain sound pulses of controlled duration [2]. Sweeping of the speed of sound in resonant acoustic layers can be used to design high-speed acoustic gates and modulators [3].

The dynamics of parametric generation depends strongly on the specific physical conditions. It is fairly diverse and interesting for not only applications but also general physics. Until now, most attention has been paid to the parametric dynamics of acoustic fields under uniform modulation of the speed parameters of medium. In this paper, we report the results of studying the generation of sound in an open acoustic resonator excited by a sound beam under nonuniform monochromatic modulation of the speed of sound in the resonator medium at the lowest resonant frequency. The expressions for the steady-state amplitudes of acoustic field in the resonator (under the conditions specified below) are obtained, and the spectral distribution and temporal profile of the output radiation are determined.

Let a flat monochromatic acoustic beam be incident on an open acoustic resonator with the length $L$. The beam is oriented along the resonator ($Z$) axis and has a frequency $\omega$, which coincides with one of the resonator eigenfrequencies: $\omega = \Omega N$, where $\Omega$ is the lowest eigenfrequency and $N$ is an integer. The resonator is completely filled with a parametrically active medium, where the speed of sound $c$ undergoes a periodic nonuniform modulation with a frequency $\Omega$ and a modulation depth $v(z) < 1$:

$$ C = C_0 \left[ 1 + v(z) \cos(\Omega t) \right]. \quad (1) $$

Note that this situation corresponds to real experimental conditions. Furthermore, we assume (without significant loss of generality) that oscillations of axial types are selected in the resonator and describe the field $U$ in the resonator in scalar terms. In this case, $\Omega \approx \pi c_0 / L$.

Taking into account the smallness of $v(z)$ for the acoustic shift $U$, we use the expansion

$$ U = \sum \left[ \frac{1}{2} Y_l(t) \exp(-i \varpi_l t) + \text{c.c.} \right] \times \sin(k_l z) g_l(\vec{r}_l). \quad (2) $$

Here, $\varpi_l = \Omega l$, $k_l = \varpi_l / c_0$, $g_l(\vec{r}_l)$ is the transverse distribution of a selected mode with the axial index $l$, and $Y_l(t)$ are slowly varying amplitudes. Substituting (2) into the wave equation for slowly varying amplitudes $Y_q$ and taking into account the orthogonality of the resonator eigenfunctions, we obtain (after averaging over fast oscillations) the equation

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\[
\frac{dY_q}{dt} = \mu_q Y_q + i \frac{\Omega^2}{\pi c_0 q} \left[ Y_{q+1}(q+1)^2(\varphi - \varphi_{q+1}) + Y_{q-1}(q-1)^2(\varphi - \varphi_{q-1}) \right] + TU_{inc} \delta_{qN},
\]
\[
\varphi \approx \frac{1}{2} \int v(z) \cos \left( \frac{\Omega}{c_0} z \right) dz \int \frac{\int g_{q+1}(\bar{r}_1) g_q(\bar{r}_1) dr_1}{g_q^2(\bar{r}_1) dr_1},
\]
\[
\varphi_{q\pm 1} \approx \frac{1}{2} \int v(z) \cos \left( \frac{\Omega}{c_0} (2q \pm 1) z \right) dz \int \frac{\int g_{q+1}(\bar{r}_1) g_q(\bar{r}_1) dr_1}{g_q^2(\bar{r}_1) dr_1}.
\]

Here, \( U_{inc} \) is the incident beam amplitude; \( T \) is the permeability coefficient of the resonator input plane for the incident beam; and \( \mu_q \approx \mu \) is the damping of resonator modes, which is phenomenologically introduced and mainly determined by the sound transmission at its faces [4].

Note that in the ultrasonic range, which is typical of sound parametric conversion, the spatial change (natural or artificial) in the modulation coefficient \( v(z) \) can be considered smooth in most cases:
\[
\frac{dv}{dz} \lesssim \frac{\Omega}{c}.
\]

Then, obviously, \( \varphi_{q\pm 1} \ll \varphi \), and the corresponding terms in the coefficients at \( Y_{q\pm 1} \) in (2) can be neglected.

Let us consider the steady-state mode, i.e., assume that \( \frac{dY_q}{dt} = 0 \) in (3). For the steady-state amplitudes, we have a system of linear inhomogeneous equations
\[
Y_q + \frac{\alpha (q+1)^2}{q} Y_{q+1} + \frac{\alpha (q-1)^2}{q} Y_{q-1} - \frac{TU_{inc}}{\mu} \delta_{qN},
\]
where
\[
\alpha = -i \frac{\Omega^2}{\pi c_0 \mu} \varphi.
\]

Furthermore, we assume that an interference coating with a resonant frequency of \( \Omega(N+1) \) is deposited on the resonator faces, then \( q \ll N \) in (5). It can be seen that the dynamics of stationary generation is determined to a great extent by the relation between the linewidth of resonator eigenoscillations and the parametric conversion coefficient of the system.

Let us consider the most interesting case of a high-Q resonator and an efficient parametric interaction, \(|\alpha| \gg 1\). For the component with the number \( q \), Cramer’s rule yields \( Y_q = M_q/A_N \), where \( A_N \) is the determinant of the fundamental matrix of the system (5) and \( M_q \) is the determinant of the matrix \( A \) with a \( q \)th column replaced by a free-term column. Expanding \( M_q \) in this column, one can easily make sure that the matrix of the cofactor of the element \( TU_{inc}/\mu \) in the columns with numbers \( n > q \) contains only zero elements that lie above the main diagonal. Accordingly,
\[
Y_q = (-1)^{N+q+1} \frac{\alpha^{N-q}}{A_N} \frac{N!N}{q!q \mu} \frac{TU_{inc}}{\mu} A_{q-1}.
\]

Here, \( A_{q-1} \) is the upper left minor of the matrix \( A \) of \((q-1)\)th order. The tridiagonal matrix \( A \) is changed by the recurrence relation
\[
A_q = A_{q-1} - \alpha^2 q(q-1) A_{q-2}.
\]

Using relation (7), one can easily find that the determinant \( A_q \) over the large parameter \( \alpha \) has orders \( \alpha^q \) and \( \alpha^{q-1} \) at even and odd \( q \), respectively. For an even \( q \), the term in \( A_q \) that corresponds to the highest power of \( \alpha \) is
\[
A_q = (-1)^{q-1} q! \alpha^q.
\]

Thus, it follows from (6) and (8) that the components generated in the system have generally odd numbers. The amplitudes of these components are
\[
Y_q = \frac{1}{A_N} \alpha^{N-1} \frac{TU_{inc}}{\mu} \frac{N!N}{q^2}, \quad q = 2n + 1.
\]

It follows from (9) that the incident radiation energy is transferred to the low-frequency range of the resonator output radiation, which leads to a temporal deformation of the sound beam. Let us find the temporal profile \( \Phi_N \) of the generated acoustic field:
\[
\Phi_N(t) = Q \sum_{n=0}^{(N-1)/2} \frac{\cos(2n+1) \Omega t}{(2n+1)^2},
\]
where \( Q \) is obviously determined from (9) and \( N \) is assumed to be odd for simplicity. The Fourier series, corresponding to sum (10), determines the even periodic function